Introduction to Spherical Astronomy

TCAA Guide #10

Carl J. Wenning

TWIN CITY AMATEUR ASTRONOMERS

ESTABLISHED 1960

Introduction to Spherical Astronomy

TCAA Guide #10

VERSION 1.3.2

NOVEMBER 26, 2020

ABOUT THIS GUIDE:

This TCAA Guide #10 – *Introduction to Spherical Astronomy* – is the tenth TCAA guide written for amateur astronomers. It was started as a book chapter in the early 1990s and didn't come to fruition until 2020 as part of the COVID-19 pandemic when the author had plenty of time on his hands to complete several of his personal bucket list items.

TCAA Guide #10 is an introduction to the basic knowledge of positional astronomy with which amateur astronomer should be familiar. While it is not a substitute for learning more extensively about the science of astronomy, this Guide provides the basic information one needs to bridge the gap from neophyte to an observer vested with the knowledge of what it takes to view the heavens using basic equipment properly. As such, this is not a reference work. It is not intended to be the answer to all questions that an amateur astronomer might have. It merely provides sufficient information to advance one in the field of amateur astronomy dealing the positions of things in the sky.

Why write *Introduction to Spherical Astronomy* when there are so many applications available for computers, cell phones, and tablets that make calculations for observational astronomy in the blink of an eye? The purpose of this Guide is not to make spherical astronomers out of amateur astronomers; rather, it is designed to give amateurs a fuller understanding of the workings of electronic planetarium applications and devices like "goto" telescopes. It also can be used to understand the methods of archeoastronomy and celestial navigation. If nothing more, this Guide will serve to make for more informed consumers of products, research, and position finding. Still, it can also provide a bit of amusement for those who would like to make at least some of these calculations on their own. Those would like additional details should turn to such works as Jean Meeus' *Astronomical Algorithms* (2nd edition, 1999, Willmann-Bell, Inc.)

The author gratefully acknowledges the assistance of Sunil Chebolu who provided the author with guidance about how best to develop some of the three dozen line drawings that appear in this edition. The author accepts all responsibility for any errors in figure or misrepresentation of fact that might appear in TCAA Guide #10.

ABOUT THE AUTHOR:

Dr. Carl J. Wenning is a well-known Central Illinois astronomy educator. He started viewing the heavens with the aid of his grandfather in the summer of 1957. Since that time, he has continued viewing the night sky for nearly six decades. He holds a B.S. degree in Astronomy from The Ohio State University, an M.A.T. degree in Planetarium Education from Michigan State University, and an Ed.D. degree in Curriculum & Instruction with a specialization in physics teaching from Illinois State University. He taught his first courses in astronomy at Alma College in 1977 and Michigan State University in 1978.

Dr. Wenning was planetarium director at Illinois State University from 1978 to 2001. From 1994-2008 he worked as a physics teacher educator. Retiring in 2008, he continued to teach physics and physics education courses for an additional twelve years. He also taught astronomy and physics lab science almost continuously at Illinois Wesleyan University from 1980 to 1999. He taught physics at Heartland Community College from 2018-2020. He now has more than 43 years of university-level teaching experience. He also taught conceptual physics at University High School during the 1994/1995 academic year.

Carl became associated with the TCAA in September 1978 – shortly after being hired to work at Illinois State University. Today he is an Astronomical League Master Observer (having completed 14 observing programs to date) and received the 2007 NCRAL Region Award for his contributions to amateur astronomy. He is a lifelong honorary member of the TCAA and is a member of its G. Weldon Schuette Society of Outstanding Amateur Astronomers. He is the 2017-2021 Chair of the North Central Region of the Astronomical League.

TABLE OF CONTENTS

Introduction to Spherical Astronomy

1. The Celestial Sphere

When we look up on a dark, clear night, we can see a myriad of stars, all apparently located on the surface of a vast hemisphere of indeterminate size. Using only our eyes, we can tell nothing about the distances of the stars. We can establish the direction of each star and can, with suitable instruments, determine the angular separations between stars with considerable precision.

An observer located in an open field has the distinct impression that he is located under a hemisphere, as shown in Figure 1. The astronomical horizon, an imaginary circle where the mean sea-level surface intersects the sky, compasses the observer. The astronomical zenith, Z, is found an angular distance of $90°$ above this horizon. It is an imaginary point on the celestial hemisphere where a line extended opposite the pull of gravity from the observer's site would intersect the celestial hemisphere.

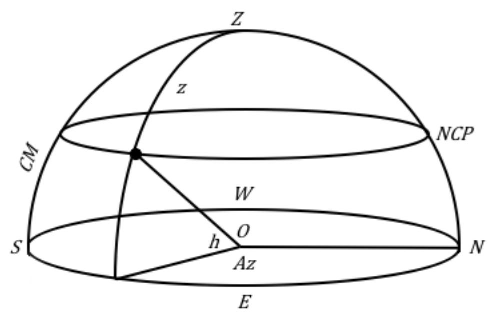


Figure 1. The celestial hemisphere with cardinal points and key locations in the sky indicated.

Cardinal points north (N), east (E), south (S), and west (W) ring the horizon, the intersection of a plane tangent to the surface of the Earth at the location of the observer with the sky. The north and south directions are uniquely defined by a line drawn tangent to the Earth at the observer's location, O, and in the plane containing Earth's rotation axis. East and west lay along a line drawn perpendicular to this north-south tangent line. Using the horizon circle with its directions and the zenith as references, it is possible to define the location of any object in the sky uniquely.

The angular distance of an object above the horizon is known at its elevation; elevation is denoted in this publication with the symbol h. An object on the horizon has an elevation of $0°$. The zenith has an elevation of $90°$. An object located halfway up in the sky has an elevation of $45°$. An object below the horizon would have an elevation of less than $0°$. Additionally, elevation is sometimes referred to as altitude. The terms elevation and altitude should not be confused with the distance of objects above ground level. Such a distance might be termed height.

Conversely, the angular distance of an object from the zenith is called zenith distance, z. An object on the horizon has a zenith distance of $90°$. Objects below the horizon would have zenith distances of greater than $90°$. Zenith distance is equal to $90° - h$ and is always positive. The nadir, the point diametrically opposite the zenith (not shown in Figure 1), has an elevation of $-90°$ and a zenith distance of $180°$.

An object located at a given angular distance above a horizon will is located upon a parallel of elevation, a circle of fixed elevation that runs parallel to the horizon. Locating an object upon a given parallel of altitude (known as an almucantar) is insufficient to describe its position uniquely. The distance along this circle also must be specified. This angular distance, measured from the north through east along the horizon, is called the azimuth (Az). Azimuth is measured from $0°$ in the north to $90°$ in the east to $180°$ in the south to $270°$ in the west and so on. An arc extended downward from the zenith through a celestial object intersects the horizon at a $90°$ angle. This line, a meridian of azimuth, is required to locate an object precisely along the horizon circle.

A particular meridian, the celestial meridian (CM), extends around the celestial sphere from north to south across the sky and passes through zenith and nadir. That portion of the celestial meridian, above the horizon, is called the upper meridian; that which is below the horizon is called the lower meridian. The celestial meridian divides the observable sky into eastern and western halves.

Lying transverse to the celestial meridian and passing through the zenith and east and west cardinal points is the prime vertical (not shown in Figure 1.) The prime vertical serves to separate the sky into northern and southern halves.

With the passing of the night, stars appear to rise in the eastern half of the sky, reach their highest elevations when they cross over or transit the upper meridian going from east to west, and then set in the western half of the sky. The observer is led to conclude by the rising and setting of stars that this hemispherical construct, which appears to hold the stars, extends below the horizon. The observer would now seem to find himself at the center of an enormous sphere of indeterminate size upon which all celestial bodies may be considered to be located.

This celestial sphere is an imaginary surface of an arbitrary radius, though presumably great, which contains the observer at its center. The celestial sphere seems to carry the stars on their appointed courses across the heavens. Lines connecting the observer and stars (which all have different distances) intersect the celestial sphere's surface. Because the distances of stars are imperceptible, it shall suffice us to assume that they are located on the surface of this celestial sphere. It is only their directions that interest us in the work of spherical astronomy.

Observations show that not all stars appear to rise and set for an observer at an intermediate northern latitude. Stars to the north remain perpetually in the sky, circling round and round a celestial pole (P) in a counterclockwise direction. Their motions seem to be nearly centered on the North Star, Polaris by name. Stars that never appear to rise or set, such as the stars of the Big and Little Dippers, Cassiopeia, Cepheus, and Draco, are called circumpolar stars.

Observers in the southern hemisphere would see a similar circumpolar motion above their southern horizon. The apparent motions of the stars would be roughly centered on the star Sigma Octantis, however, and their motions would be in clockwise rather than in a counterclockwise direction.

A northern hemisphere observer can identify a point above the northern horizon where stars, even the North Star, appear to circle. The north celestial pole (NCP) is the point on the celestial sphere determined by an extension of Earth's northern rotational axis to the point where it reaches the celestial sphere. Similarly, the southern hemisphere observer would find the south celestial pole (SCP) above his southern horizon where stars, even Sigma Octantis, would appear to circle.

Outside the zone of circumpolar stars, an observer would find stars that do appear to rise and set. These stars are known as equatorial stars. The stars of Pegasus, Orion, Leo, and Cygnus are among this group for observers in the mid-northern latitudes.

Circumpolar stars transit the celestial meridian twice every time the Earth rotates through 360° – once above the celestial pole and once below the celestial pole. When circumpolar stars transit the upper meridian and reach their highest altitudes above the horizon, they are said to be at upper culmination. When circumpolar stars transit the upper meridian and reach their lowest altitudes, they are said to be at lower culmination.

Similarly, equatorial stars transit the meridian twice every time the Earth completes one rotation with respect to the stars – once at the upper meridian and once at the lower meridian. Such transits are termed upper transit and lower transit, respectively.

The relationship of the north celestial pole to planet earth is shown in Figure 2. The north celestial pole stands directly over the north terrestrial pole. Similarly, the south celestial pole stands directly over the south terrestrial pole. An astute northern hemisphere observer might also note that there is a region of the celestial sphere whose stars lie below the southern horizon and never come into view. Such star patterns for our mid-northern latitude observer would include the Southern Cross, Centaurus, the Magellanic Clouds, and so on. Similarly, observers in the mid-southern latitudes would never be able to see the Big and Little Dippers or Cassiopeia.

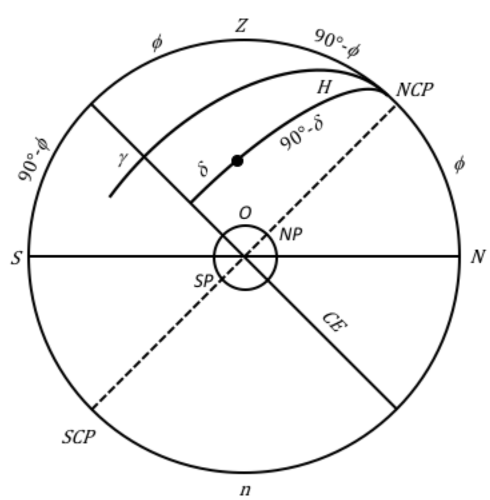


Figure 2. Relationship of terrestrial and celestial coordinate systems.

The terrestrial equator is defined by a plane drawn perpendicular to the Earth's rotation axis and located midway between its poles. An extension of this plane into space intersects with the celestial sphere to produce the celestial equator (CE), a circle $90°$ from both celestial poles, as seen by an Earthbound observer. The celestial equator spans the sky and intersects the horizon at the east and west cardinal points.

A circle on the celestial sphere that passes through the celestial poles is perpendicular to the celestial equator. Such a north-south circle on the celestial sphere is known as an hour circle. Hour circles are analogous to meridians of longitude on Earth. The angular distance measured eastward along the celestial equator is called Right Ascension, denoted with the Greek letter alpha (α). Unlike longitude, which is measured from the prime meridian, right ascension is measured from a point on the celestial equator known as the First Point of Aries. The First Point of Aries represents the intersection of the sun's hour circle with the celestial equator on the date of the March equinox. This hour circle is commonly referred to as the equinoctial colure; by analogy, it is the prime meridian of the heavens, though it is never referred to as such. The First Point of Aries is represented by the Greek letter gamma (γ). The right ascension of the First Point of Aries is, by definition, zero hours.

Right ascension is measured continuously eastward along the celestial equator from the First Point of Aries to the hour circle passing through the celestial object. Traditionally, right ascension is measured in hours, minutes, and seconds, where 15° of arc equal one hour of right ascension on the celestial equator. The relationship between degrees (°), minutes ('), and seconds (") of arc and hours (h), minutes (m), and seconds (s) of right ascension as is follows:

24 hours = 360°; 1h = 15°; 1m = 15'; and 1s = 15"

Like the Earth's coordinate system of latitude, the angular distance of a celestial object north or south of the celestial equator measured along the hour circle passing through the object is called declination. It is denoted with the Greek letter delta (δ). North declinations are taken as positive; south declinations as negative. The celestial equator's declination is $0°$; the declinations of the north and south celestial poles are taken as $+90°$ and $-90°$, respectively.

The meridian altitude of an observer's celestial pole depends upon latitude. The relationships between these two and the observer's latitude can be derived from a study of Figure 3. The observer is located at O on the Earth (located in the center of the celestial sphere and assumed perfectly spherical and miniscule with respect to the size of the celestial sphere) which has its center at C and an equator which when extended to the celestial sphere forms $\overline{EQ}$. Directly above the terrestrial pole lies the celestial pole. Line $\overline{CP}$ extends up toward the celestial pole, P. Because the celestial sphere is of vast size compared to the Earth, line $\overline{ON}$ points to the north celestial pole and is parallel to $\overline{CP}$. Now, $\widehat{HON} = h_{cp}$. Also, $\widehat{OCQ} = \phi$, the observer's latitude, and $\widehat{HOC} = 90°$. Because $h_{cp} + 90° + (90° - \phi) = 180°,\ h_{cp} = \phi$.

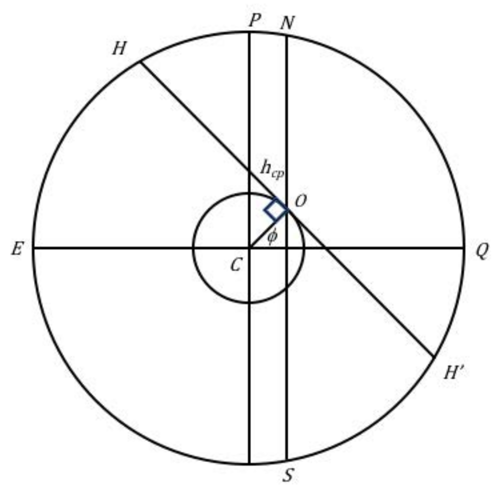


Figure 3. Relationship between the observer's latitude and the elevation of the celestial pole.

Because the altitude of the celestial pole is equal to the observer's latitude, an observer at the equator (latitude 0°) would see both celestial poles on the horizon; an observer at either pole (latitude 90° north or south) would see the corresponding celestial pole at zenith; an observer located midway between the pole and the horizon would see the celestial pole halfway up in the sky.

The meridian altitude of the celestial equator can be found by referring to Figure 4. $\overline{NS}$ marks the horizon centered on the observer at O. Z marks the zenith, P the observer's celestial pole, and CE the celestial equator. Now, the angular distance of the poles from the celestial equator is 90°. The elevation of the pole equals the observer's latitude, ϕ. $\widehat{SOC}$ is the elevation of the celestial equator, θ, at the observer meridian. Because $\widehat{SOC} + \widehat{COP} + \widehat{PON} = 180°$, we can conclude that the meridian altitude of the celestial equator equals $90° - \phi$. Hence, at either of Earth's poles, the celestial equator runs parallel to the horizon; at the equator, it passes through the zenith.

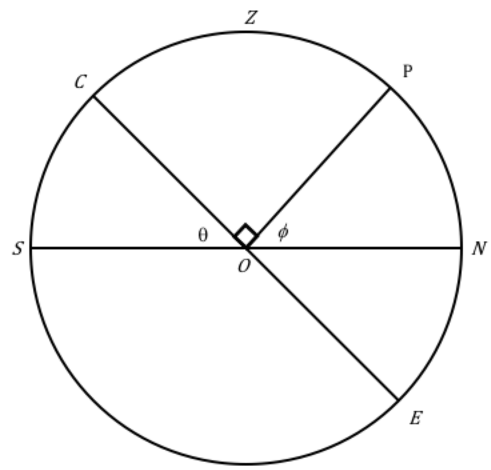


Figure 4. The relationship between the observer's latitude and the elevation of the celestial equator at the meridian.

With these reference points and circles defined and the relationship between the latitude of the observer and the altitudes of the celestial poles and equator known, it is possible to describe more completely the positions and motions of objects on the celestial sphere with respect to the observer, the observer's horizon and zenith, or the celestial sphere's equator and poles.

2. Introduction to Spherical Astronomy

Spherical astronomy is primarily a study of the directions of celestial objects as seen from any position on the surface of the Earth. These directions can be measured conveniently with respect to points on the Earth or in the heavens. The study of spherical astronomy is fundamental to any astronomer's understanding of the motions of the heavens. Spherical astronomy is essentially a study of the mathematics of the celestial sphere.

In this Guide, the reader learns about the fundamental mathematical formulae associated with the study of the celestial sphere and see numerous examples of their applications. To assist in such learning, two devices are available for study. The celestial sphere has been replicated in the form of celestial globes of various makes and models and one is shown in Figure 5a. This provides for an external view of the celestial sphere. Seen from the outside, constellations are displayed in reverse. A planetarium, as shown in Figure 5b, also is a device that replicates a celestial hemisphere as viewed from within. Inside a planetarium, constellations appear as though they would in the night sky.

Figures 5a & 5b. Transparent celestial globe and a conventional planetarium with a hemispheric dome.

2a. Initial Results

From the finding $h_{CP} = \phi$, we know that the altitude of the celestial pole is equal to an observer's latitude. Given this fact, we can draw a celestial sphere as shown in Figure 6. Here we see $\widehat{PON} = \widetilde{PN}$ because $\widehat{PON}$ is 90° away from $\widetilde{PN}$.

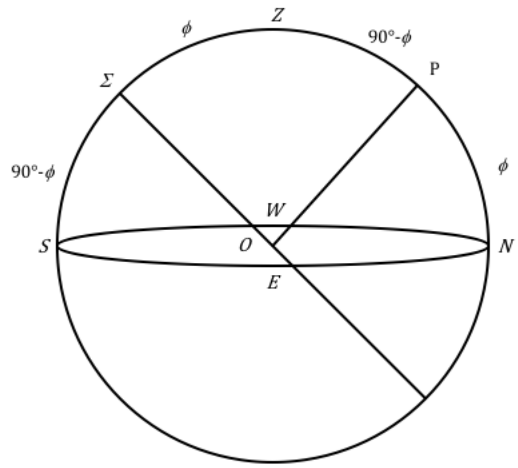


Figure 6. Location of various celestial points relative to the observer at O as well as the horizon.

Let N, E, S, and W mark the cardinal points on the observer's horizon. Z marks the zenith, P the position of the North Celestial Pole, and Σ, the **Sigma Point,** the intersection of the celestial equator and celestial meridian. The observer is located at the center of the celestial sphere at O.

Now, $\widehat{NOP}$ is equal to the observer's latitude, which is designated ϕ. Because $\widehat{NOZ}$ equals 90°, $\widehat{POZ}$ equals $90° - \phi$. Similarly, $\widehat{ZO\Sigma} = \phi$ and $\widehat{\Sigma OS} = 90° - \phi$.

Given that the distance measured perpendicular from the celestial equator is the declination, δ, the following relationships can be derived for points on the celestial meridian:

1. The declination of the zenith is equal to the observer's latitude, ϕ.
2. A star is circumpolar if $\delta > 90^{\circ} - \phi$
3. When an equatorial star transits the meridian, the altitude of the star equals $90° + \delta - \phi$.
4. A circumpolar star at upper culmination transits the celestial meridian at an altitude of $\phi - \delta + 90°$.
5. A circumpolar star at lower culmination transits the celestial meridian with an altitude of $\phi + \delta - 90°$.
6. The altitude of the sigma point equals the observer's **co-latitude**, $90^{\circ} - \phi$.
7. The declination of the most southerly star theoretically visible from latitude ϕ equals $\phi - 90°$.
8. The declination of the nadir equals $-\phi$.

By inspection, we also can obtain two additional results for points along the horizon. Because the celestial equator is everywhere 90° away from the celestial pole:

9. The azimuths where the celestial equator intersects the horizon equals 90° and 270°.
10. The angle that the celestial equator makes with the horizon equals $90° - \phi$.

These results are limited to a few points on the celestial meridian and the horizon. If we are to come up with more general results – equations that deal with points not solely on the observer's celestial meridian and horizon – then we must begin studying spherical trigonometry in earnest. But first, something must be said about the non-spherical nature of Earth and the consequence of this non-sphericity on positions of objects in the sky.

2b. The Geoid

The construction of a celestial sphere is simple and very useful. In this discussion, the Earth has been assumed to be perfectly spherical even though that is not the case. Errors arising from the assumption are minimal for naked-eye astronomy and were therefore ignored. More precise calculations using horizon-based coordinates must take the non-spherical form of the Earth into account.

Earth is a large, self-gravitating body. Such bodies, if sufficiently fluid, form perfect spheres when non-rotating. Because the Earth rotates, its shape is not that of a perfect sphere but rather a spheroid of revolution. This spheroid of revolution is more appropriately referred to as the **standard geoid**. It represents an imaginary equilibrium surface defined by the mean sea-level surface of the Earth. It is this mean sea-level surface that defines the horizon and astronomical zenith of an observer.

Figure 7 represents a highly exaggerated view of the standard geoid. In reality, the polar radius of the Earth is only $22km$ less than the equatorial radius of $6{,}371km$. Consider an observer located at O on the surface of the geoid that has its center at C. A line extended upward from the center of the Earth through the observer points to the **geocentric zenith** indicated by Z'. The **geocentric latitude** of the observer is represented by ϕ'. A line extended upward from our observer and perpendicular to a plane tangent to the geoid (the observer's horizon) intersects a point on the celestial sphere known as the **astronomical zenith** labeled Z. This line extended down to the equatorial radius forms an angle ϕ, the **astronomical latitude**.

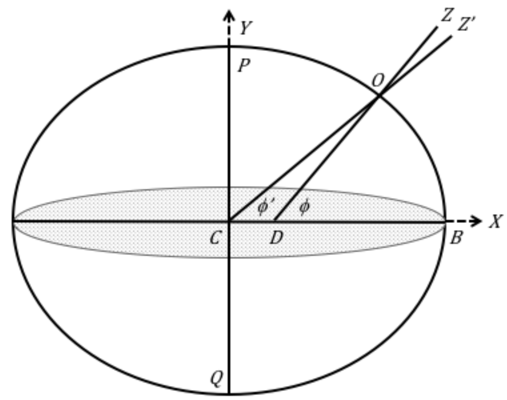


Figure 7. The Earth geoid greatly exaggerated to show the difference between geocentric and astronomical zeniths.

The difference between angle ϕ and ϕ' is called the **angle of the vertical**. It is equal in magnitude to the difference between Z and Z'. The magnitude of difference can be written as a function of astronomical latitude approximately as follows:

$$v = \phi - \phi' = 695.65'' \sin 2\phi - 1.17'' \sin 4\phi$$

The non-spherical nature of Earth shifts the geocentric zenith from the astronomical zenith in the direction of the Earth's equator by the amount v. At the equator and the poles, the astronomical and geocentric zeniths coincide. Their divergence is at a maximum at approximately $44.9°$ north and south latitude. It amounts to about 12" or one-fifth of one degree.

Generally speaking, calculations made by amateur astronomers need not be of the highest order of precision. If the complications introduced by the shape of the Earth are ignored, errors due to the angle of the vertical never amount to more than two-tenths of a degree. The distinction between astronomical and geocentric zeniths must be taken into account only if a high order of precision is needed. This is quickly done by substituting the geocentric latitude, ϕ', of an observing site with its astronomical latitude. The astronomical latitude, ϕ, can be found by the application of the above equation.

Additionally, several other small but essential corrections such as refraction and parallax must also be applied to calculated altitude values precisely. Because a high level of precision is rarely required for amateur work, future distinctions between astronomical and geocentric zenith are ignored here, and the Earth is treated as though it is perfectly spherical. However, adequate detail is provided in this work so that anyone wishing to produce calculations of higher accuracy can do so.

2c. Great and Small Circles

Any plane passing through the center of a sphere intersects the sphere producing a **great circle**. Any other plane intersecting the sphere but not passing through the center cuts the sphere in what is known as a **small circle**. Each circle has characteristics that must be examined in detail before discussing the spherical triangle.

Consider the sphere shown in Figure 8. Let $\breve{EQ}$ be an arc of a great circle centered on C. Let $\breve{AB}$ be an arc of a small circle centered on D and part of the plane parallel to the plane containing points C, E, and Q. Now, $\breve{AB} = DA\ x\ \widehat{ADB}$ when the angle is expressed in radian measure. Similarly, $\breve{EQ} = CE\ x\ \widehat{ECQ}$.

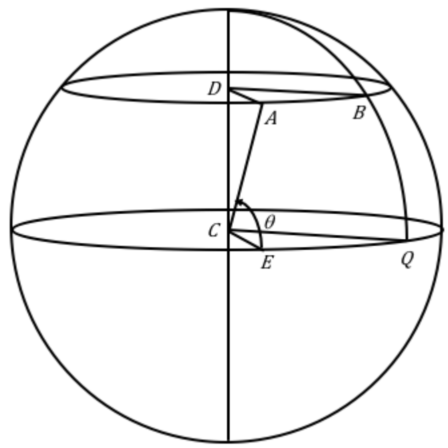


Figure 8. The relationship between great and small circles.

Because DA is parallel to CE and DB is parallel to CQ, $\widehat{ADB}$ equals $\widehat{ECQ}$. Solving each of these equations for their angles and equating yields:

$$\frac{AB}{DA} = \frac{EQ}{CE}$$

or rewriting,

$$\frac{AB}{EQ} = \frac{DA}{CE}$$

Now, CE equals CA as both are radii of the sphere. Hence this relationship can be rewritten as

$$\frac{AB}{EQ} = \frac{DA}{CA}$$

Let $\widehat{ECA} = \theta$ whence $\widehat{ACD} = 90° - \theta$. Because $sin\ (90° - \theta) = cos\ \theta = DA/CA$, we have $AB/EQ = cos\theta$. Therefore, we arrive at a critical result

$$AB = EQ\ cos\ \theta.$$

This indicates that a small circle arc converges as the plane containing it departs farther and farther from the center of the sphere. Two stars separated by, say, 10° of azimuth appear much farther apart if near the horizon than another pair separated by the same number of degrees of azimuth located higher up in the sky. Similarly, two stars separated by a given amount in right ascension are closer together if near the celestial poles than near the celestial equator. This complication also applies to two places on Earth separated by a given longitude difference. Because meridians converge as they approach the poles of a sphere, small circles generally prove to be unsuitable for a detailed study of the celestial and terrestrial spheres.

The geometry of terrestrial and celestial spheres is greatly simplified by the use of **great circles** (circles whose planes pass through the center of terrestrial or celestial spheres). The angle subtended from the center of a sphere equals the arc length on the surface of the sphere. Fortunately, right ascension, declination, the celestial equator, altitude, azimuth, and the horizon are, by definition, great circles or segments thereof. **Additionally, the lengths of great circle arcs are equal to the angles at their poles.** This makes great circle arcs especially useful in the study of spherical astronomy.

2d. The Spherical Triangle

Let O represent the apparent place of an object located "on" the celestial sphere but not on the observer's celestial meridian. Such a situation is shown in Figure 9.

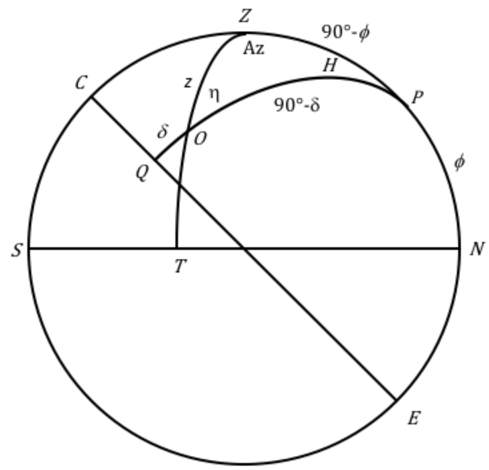


Figure 9. A spherical triangle, PZO, showing the relationship of object at O relative to other celestial angles.

The zenith is located at Z, and the horizon is indicated by the line STN. The celestial pole is located at P; the celestial equator is indicated by the line CE. The altitude of the celestial pole is ϕ; therefore, the arc ZP equals $90° - \phi$.

A great circle arc is drawn from Z through O intersecting the horizon at T. The altitude of O, given by h, is the arc TO. The zenith distance, z, therefore, equals $90° - h$. Similarly, a great circle arc connecting P and O and extended to the celestial equator intersecting it at Q. QO is the declination, δ, of the object at O. The arc OP is, therefore, equal to $90° - \delta$.

$\widehat{PZO}$, measured from the north through east, is the angular distance of the object along the horizon. $\widehat{PZO}$ is, therefore, the azimuth, Az, of the object in question. $\widehat{ZOP}$ is frequently referred to as the **parallactic angle**.

The remaining angle, $\widehat{OPZ}$, is called the **hour angle** of the object, H. It represents the angular distance on the celestial sphere measured along the celestial equator from the celestial meridian to the hour circle that passes through the celestial object in question. Hour angle traditionally is measured in hours or the degree equivalent. It is measured *westward* from the celestial meridian up to but not including 24 hours or 360°. *If measured eastward, H is considered negative.*

The great circle arcs connecting the zenith, celestial pole, and object, as indicated above, constitute a **spherical triangle**. The astronomer can solve the relationships between the sides and angles of the spherical triangle as handily as the geometer does the plane triangle. The spherical triangle solution forms the basis of **spherical trigonometry**, a study of the relationships between the angles and sides of the spherical triangle.

3. Fundamental Formulas

There are four fundamental formulas associated with the study of spherical astronomy. Each of the four is especially helpful for solving the individual parts of spherical triangles in relation to different angles and sides. The derivation of each formula is as follows.

3a. The Cosine Formula

Let O be the center of the celestial sphere, as shown in Figure 10. Let $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ represent unit vectors in the x, y, and z directions, respectively. Let $\boldsymbol{R}$ be an arbitrary *unit* vector indicating the direction of an object in space as seen from O. Let the angle between $\boldsymbol{R}$ and $\boldsymbol{k}$ be called θ. The projection of $\boldsymbol{R}$ onto the xy plane forms the line ℓ. Let the angle between ℓ and $\boldsymbol{i}$ be designated ψ.

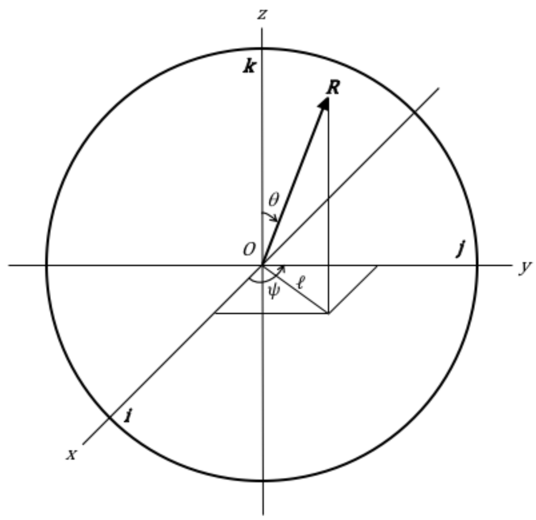


Figure 10. Polar coordinates of vector $\boldsymbol{R}$ oriented in a random direction.

From our knowledge of plane trigonometry, we can write the rectangular coordinates (x, y, z) of the object on the celestial sphere in the form of spherical-polar coordinates as follows:

$$x = sin\ \theta\ cos\ \psi$$

$$y = sin\ \theta\ sin\ \psi$$

$$z = cos\ \theta$$

The vector components of $\boldsymbol{R}$ can then be written as:

$$\boldsymbol{R} = sin\ \theta\ cos\ \psi\ \boldsymbol{i}, sin\ \theta\ sin\ \psi\ \boldsymbol{j}, cos\ \theta\ \boldsymbol{k}$$

or

$$\boldsymbol{R} = (sin\ \theta\ cos\ \psi, sin\ \theta\ sin\ \psi\ cos\ \theta)$$

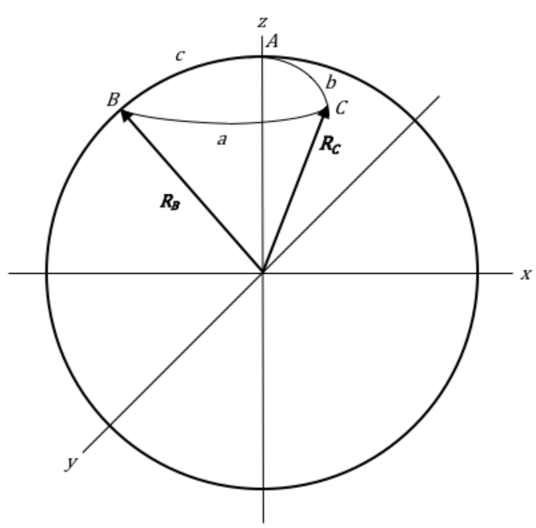


Figure 11. Forming the dot-product of three vectors results in the cosine formula.

In Figure 11, C represents an arbitrary point on the celestial sphere; $\boldsymbol{R}_C$ represents a unit vector in the direction of C. Similarly, B is a point in the xz reference plane, and $\boldsymbol{R}_B$ is the unit vector in the direction of B. Then,

$$\boldsymbol{R}_B = (sin\ c, 0, cos\ c)$$

and

$$\boldsymbol{R}_C = (sin\ b\ cos\ A, sin\ b\ sin\ A, cos\ b)$$

where b, c, and A are arcs and angle as indicated in the spherical triangle shown.

The **scalar** or **dot product** of $\boldsymbol{R}_B$ and $\boldsymbol{R}_C$ is, by definition, equal to the magnitude of each vector times the cosine of the angle between the two vectors. That is,

$$\boldsymbol{R}_B \cdot \boldsymbol{R}_C = |\boldsymbol{R}_B|\,|\boldsymbol{R}_C|\,cos\ a = cos\ a$$

because $|\boldsymbol{R}_B| = |\boldsymbol{R}_C| = 1$.

The scalar product of two vectors is also the sum of the products of the individual terms of the two vectors.

$$\boldsymbol{R}_B \cdot \boldsymbol{R}_C = sin\ b\ cos\ A\ sin\ c + cos\ b\ cos\ c$$

Equating the results of the two dot products and rearranging terms leads to the first and most fundamental formula of spherical astronomy:

$$cos\ a = cos\ b\ cos\ c + sin\ b\ sin\ c\ cos\ A$$

This formula is known as the **cosine formula**. It is very useful when two sides and an included angle are known, and the third side is desired or when three sides are known, and an included angle is desired.

This formula can be generalized through the permutation of terms because of the symmetry of the spherical triangle. That is,

$$cos\ b = cos\ c\ cos\ a + sin\ c\ sin\ a\ cos\ B$$

$$cos\ c = cos\ a\ cos\ b + sin\ a\ sin\ b\ cos\ C$$

The **principle of duality** for spherical triangles permits the replacement of each side by the supplement of the opposite angle ($e.g.$, a is replaced by $180° - A$) and each angle by the supplement of the opposite side (i.e., 180° -a replaces a). Hence, the cosine formula can be rewritten as:

$$-cos\ A = cos\ B\ cos\ C - sin\ B\ sin\ C\ cos\ a$$

This equation is sometimes called the **polar version of the cosine formula** and, like the cosine formula, has similar permutations:

$$-cos\ B = cos\ C\ cos\ A - sin\ C\ sin\ A\ cos\ b$$

$$-cos\ C = cos\ A\ cos\ B - sin\ A\ sin\ B\ cos\ c$$

These equations are very helpful when two angles and an included side are known, and the third angle is desired or when three angles are known, and a side is desired.

3b. The Sine Formula

Additional results can be derived from an analysis of the **vector** or **cross product** of vectors $\boldsymbol{R}_B$ and $\boldsymbol{R}_C$. From the definition of the cross product, we can write:

$$\boldsymbol{R}_C \times \boldsymbol{R}_B = sin\ a\ \boldsymbol{R}_D$$

where $\boldsymbol{R}_D$ is a vector perpendicular to the plane containing both $\boldsymbol{R}_B$ and $\boldsymbol{R}_C$. $\boldsymbol{R}_D$ points to D on the celestial sphere, as shown in Figure 12.

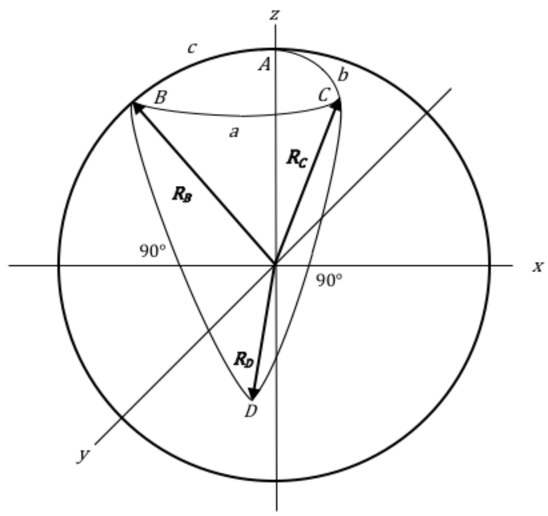


Figure 12. Forming the cross-product of three vectors results in the sine formula.

From the study of vectors, we know that $\boldsymbol{R}_C \times \boldsymbol{R}_B$ can be written in determinant form as:

$$\boldsymbol{R}_C \times \boldsymbol{R}_B = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \sin b \cos A & \sin b \sin A & \cos b \\ \sin c & 0 & \cos c \end{vmatrix}$$

which yields:

$$\boldsymbol{R}_C \times \boldsymbol{R}_B = (sin\ b\ sin\ A\ cos\ c)\boldsymbol{i} + (cos\ b\ sin\ c - sin\ b\ cos\ A\ cos\ c)\boldsymbol{j} + (-sin\ b\ sin\ A\ sin\ c)\boldsymbol{k}$$

From an examination of Figure 10, we can see that the components of $\boldsymbol{R}_D$ can be written as follows:

$$\boldsymbol{R}_D = (sin\ AD\ cos\ \widehat{BAD}\ \boldsymbol{i} + sin\ AD\ sin\ \widehat{BAD}\ \boldsymbol{j} + cos\ AD\ \boldsymbol{k})$$

The components of $\boldsymbol{R}_C \times \boldsymbol{R}_B$ should equal the corresponding components of $\boldsymbol{R}_D$ when multiplied by the term $sin\ a$. In the case of the $\boldsymbol{k}$ components, we have the following:

$$-sin\ b\ sin\ A\ sin\ c = sin\ a\ cos\ AD$$

Consider the spherical triangle ACD in Figure 10. Because $\boldsymbol{R}_D$ is perpendicular to $\boldsymbol{R}_C \times \boldsymbol{R}_B$, CD equals 90°. Therefore $\widehat{ACD} = 90° + B$. Applying the cosine formula to triangle ACD yields:

$$cos\ AD = cos\ c\ cos\ 90° + sin\ c\ sin\ 90°\ cos\ (90° + B)$$

Therefore

$$cos\ AD = -sin\ c\ sin\ B.$$

Substituting this result into the terms of the $\boldsymbol{k}$ components yields:

$$-sin\ b\ sin\ A\ sin\ c = sin\ a\ (-sin\ c\ sin\ B)$$

or, after simplification,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}$$

Symmetry allows for a more complete description of the results.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

This relationship is commonly known as the **sine formula**. It is a fundamental equation of spherical astronomy. It is useful for finding opposing sides and angles. It must be used with caution because the relationship suffers from an inherent defect of ambiguity whenever an angle or an arc is greater than 90°.

3c. The Analogue Formula

Applying the new-found sine formula to the spherical triangle ABD of Figure 11 yields:

$$\frac{\sin \widehat{ABD}}{\sin AD} = \frac{\sin \widehat{BAD}}{\sin BD}$$

or

$$sin\ AD\ sin\ \widehat{BAD} = sin\ BD\ sin\ A\widehat{B}D = sin\ 90°\ sin\ (90° + B) = cos\ B$$

Substitution of this result into the terms of the $\boldsymbol{j}$ component of $\boldsymbol{R}_D$ multiplied by $sin\ a$ and equating terms of the components yields:

$$sin\ a\ cos\ B = cos\ c\ sin\ b - sin\ c\ cos\ b\ cos\ A$$

This result is known as the **analog formula**. There are six variations of this formula, considering the permutation of terms and the dual nature of all spherical astronomy formulas.

3d. The Four-Parts Formula

Further consideration of the $\boldsymbol{i}$ component of $\boldsymbol{R}_C \times \boldsymbol{R}_B$ vector leads to a restatement of the sine formula. One final approach is taken to derive the fourth and final fundamental formula of spherical astronomy. Write the cosine formula as:

$$cos\ b = cos\ a\ cos\ c + sin\ a\ sin\ c\ cos\ B$$

Now, using the cosine formula, we can rewrite $cos\ c$ as:

$$cos\ c = cos\ a\ cos\ b + sin\ a\ sin\ b\ cos\ C$$

and with the aid of the sine formula, we can rewrite $sin\ c$ as:

$$sin\ c = \frac{sin\ C\ sin\ B}{sin\ b}$$

Making these substitutions and using the well-known trigonometric identity $cos^2 a = 1 - sin^2 a$, the following equation results after simplification:

$$cos\ a\ cos\ C = sin\ a\ cot\ b - sin\ C\ cot\ B$$

This formula is known as the **four-parts formula**. Six variations of this formula are possible considering the permutation of terms and the duality of spherical formulas. Because the sequence of terms does not lend itself to ready memorization, the equation can be rewritten as follows:

$$cos\ (inner\ side)\ cos\ (inner\ angle) = sin\ (inner\ side)\ cot\ (other\ side) - sin\ (inner\ angle)\ cot\ (other\ angle)$$

The four-parts formula is particularly well suited to relating four consecutive parts of a spherical triangle, two sides, and two angles.

These four fundamental formulas and their variations can be used to solve almost any conceivable problem dealing with the positions of objects on the celestial sphere. Just how these formulas can be used to advantage is shown in the following section.

4. Numerical Examples

The utility of these formulas, as applied to either the celestial or terrestrial sphere, can be shown through examples. What follows are several examples that show the utilization of the standard formulas.

EXAMPLE 1:

A jet airliner departs Chicago for London. What is the air distance of the great circle route? What is the azimuth of departure? What is the greatest northerly latitude reached?

Consider the spherical triangle stretching between Chicago (C), London (L), and the geographic North Pole (P), as shown in Figure 13. Arc CL is the air distance, d; CP and LP are the co-latitudes, and $\widehat{CPL}$ is the difference in longitude, $\Delta\lambda$.

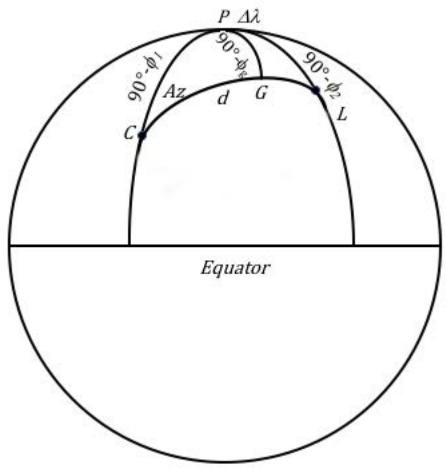


Figure 13. Spherical triangles related to the air route flown by a jet airliner going from Chicago to London.

The longitude (angular distance east or west of the Earth's **reference meridian** or prime meridian) and latitude (angular distance north or south of the equator) of each city are given as follows:

	Chicago	London
Longitude	$\lambda_1 = -87°\,45'$	$\lambda_2 = +00°\,10'$
Latitude	$\phi_1 = +41°\,50'$	$\phi_2 = +51°\,30'$

Note the negative sign on λ_1. By modern convention, measurements of longitude east of the prime meridian are considered positive. Measurements made to the west are therefore considered negative. Hence, the difference in longitude, $\Delta\lambda$, equals $\lambda_2 - \lambda_1$ or $87°\,55'$. North latitudes are considered positive, whereas south latitudes are considered negative.

Applying the cosine formula to spherical triangle CPL results in the formulation:

$$cos\ d = cos\ (90° - \phi_1)\ \cos\ (90° - \phi_2) + \sin\ (90° - \phi_1)\ \sin\ (90° - \phi_2)\ \cos\Delta\lambda$$

that reduces to:

$$cos\ d = sin\ \phi_1\ sin\ \phi_2 + cos\ \phi_1\ cos\ \phi_2\ cos\ \Delta\lambda$$

because $sin\ (90° - \theta) = cos\ \theta$ and $cos\ (90° - \theta) = sin\ \theta$.

Inserting the values and solving the equation for d results in an air distance of $57.4°$ of arc. Because one minute of arc on the surface of the Earth equals approximately $1.853 km$ ($360°$ at the equator corresponding to the circumference of the Earth at the equator, $40{,}030 km$), the total air distance is $6{,}380 km$.

The azimuth of departure from Chicago, Az, is given by $\widehat{PCL}$. Once again applying the cosine formula to the triangle yields:

$$cos\ (90° - \phi_2) = cos\ (90° - \phi_1)\ cos\ d + sin\ (90° - \phi_1)\ sin\ d\ cos\ Az$$

$$cos\,Az = \frac{sin\,\phi_2 - sin\,\phi_1\;cos\,d}{cos\,\phi_1\;sin\,d}$$

Solving for Az and inserting all variables results in an angle of departure equal to $47.6°$, or roughly northeast.

The greatest northerly latitude reached occurs when the airliner flies along a parallel of latitude and perpendicular to a meridian of longitude. This occurs at point G along the great circle route. Construct the line PG perpendicular to CL. The length of PG is $90° - \phi_g$ where ϕ_g is the greatest northerly latitude reached. The sine formula can be used to advantage to solve for ϕ_g.

$$\frac{sin\,90°}{sin\;(90° - \phi_1)} = \frac{sin\,Az}{sin\left(90° - \phi_g\right)}$$

$$cos\,\phi_g = sin\,Az\;\;cos\,\phi_1$$

Inserting the values and solving for ϕ_g yields $56.6°$ north latitude, a value of latitude higher than London's. Under such circumstances, the airliner approaches London from the north of west. The precise angle can be readily calculated using the cosine formula.

EXAMPLE 2:

Given the sun's right ascension, α, determine its declination, δ, using knowledge of the fact that the sun moves along the ecliptic, which intersects the celestial equator at a known angle, ε, the obliquity of the ecliptic, as shown in Figure 14.

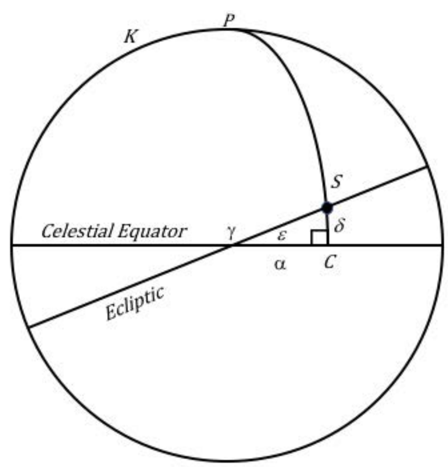


Figure 14. Relationship between the ecliptic and celestial equator.

Let P represent the celestial pole and K the ecliptic pole. Let S represent the position of the sun on the ecliptic. The celestial equator and the ecliptic intersect at the point γ, the position of the March equinox. The angle between the two, ε, has a magnitude of about 23.44°.

Now SC is the sun's declination, δ. γC is the sun's right ascension, α. SC is perpendicular to γC. Applying the four-parts formula to spherical triangle $SC\gamma$ gives:

$$cos\,\alpha\;cos\,90° = sin\,\alpha\;cot\,\delta - sin\,90°\;cot\,\varepsilon$$

$$sin\,\alpha\;cot\,\delta = cot\,\varepsilon$$

$$tan\,\delta = sin\,\alpha\;tan\,\varepsilon$$

Hence, knowledge of the sun's right ascension is sufficient to determine its declination. This result proves useful in examples to come.

EXAMPLE 3:

What is the hour angle, H, of an object rising due east? That is, with an azimuth, Az, of $90°$? Consider the solution of these problems with the use of Figure 15.

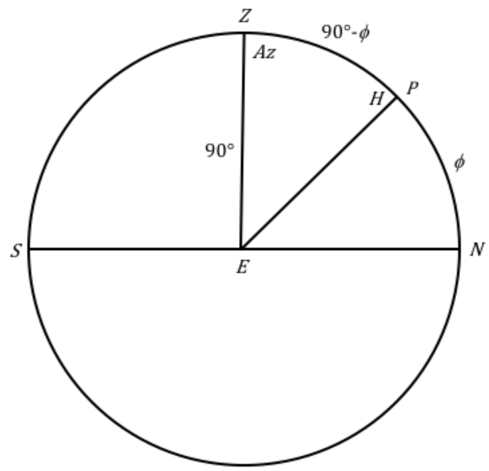


Figure 15. The hour angle, H, for an object rising due east with an azimuth of 90° from latitude ϕ.

A similar application of the four-parts formula to Figure 14 yields:

$$cos\ (90° - \phi)\ cos\ Az = sin\ (90° - \phi)\ cot\ 90° - sin\ Az\ cot\ H$$

or

with a valuable result:

$$H = 90°$$

EXAMPLE 4:

One can readily determine the approximate length of day (LoD) from a knowledge of the hour angle of rising. The LoD is approximate 2/15 this value; the 2 coming from the LoD being twice the length of time for the sun to move from the horizon to the meridian and the 1/15, which is required to convert the hour angle of rising into hours of time. Consider Figure 16.

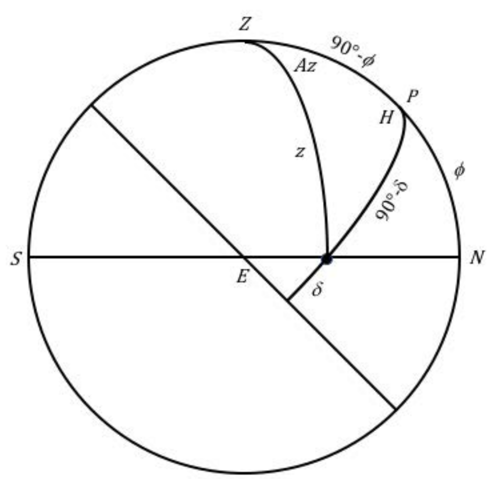


Figure 16. The sun's position at rising, including zenith distance (90°) and hour angle.

Calculate the hour angle of rising (in degrees) using the cosine formula with the spherical triangle shown in Figure 15

$$cos\ z = cos(90° - \phi)\ cos(90° - \delta) + sin(90° - \phi)\ sin(90° - \delta)\ cos\ H$$

Now, if z is taken to be 90° (a rough approximation of the sun's zenith distance at rising), the relationship simplifies to

$$0 = sin\,\phi\, sin\,\delta + cos\,\phi\, cos\,\delta\, cos\,H$$

or

$$cos\,H = -\,tan\,\phi\, tan\,\delta$$

$$LoD = 2/15\,H = 2/15\,cos^{-1}\,(-tan\ \phi\ tan\ \delta)$$

Sunrise occurs when the sun's center is located a distance below the horizon equal to its semidiameter (16') plus a correction for refraction of about 34'. Ergo, sunrise occurs when the sun's zenith distance $z = 90°\,51' = 90.85°$. Thus, a better value for the length of the day (though still not precise due to the sun's motion along the ecliptic) can be found by using the following formula. The proof of the following equation is left to the reader.

$$LoD = \frac{2}{15} cos^{-1}((cos\,90.85° - sin\ \phi\ sin\ \delta)/(cos\ \phi\ cos\ \delta))$$

The sun's hour angle at rising or setting is given by the following relationship. Keep in mind that $H < 0$ in the event of rising and $H > 0$ in the event of setting.

$$H = \frac{1}{15} cos^{-1}((cos\,90.85° - sin\ \phi\ sin\ \delta)/(cos\ \phi\ cos\ \delta))$$

EXAMPLE 5:

We want to calculate the altitude, h, and azimuth, Az, of a star located in the western sky at some particular point in time. The pertinent application of spherical trigonometry to the celestial sphere is shown in Figure 17.

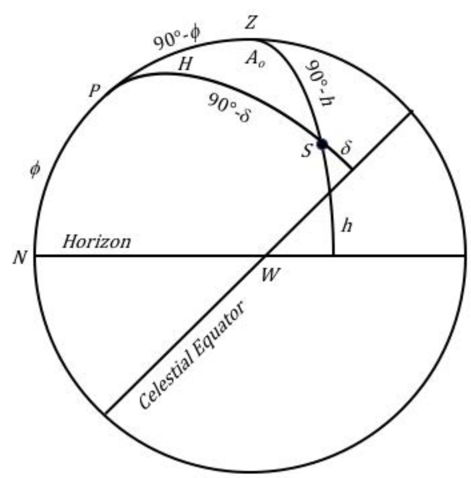


Figure 17. Sun's hour angle and auxiliary angle for a given position in the western sky.

Let S represent the position of a star on the surface of the celestial sphere. Z marks the zenith; P the position of the north celestial pole. Because the altitude of the pole is equal to the observer's latitude, PZ is the co-latitude, $90° - \phi$. PS is equal to 90° minus the declination of the star or $90° - \delta$. The zenith distance, ZS, equals 90° minus the altitude or $90° - h$. $\widehat{ZPS}$ is the hour angle of the star, H. $\widehat{PZS}$ is called A_o, the **auxiliary angle** which is measured from the north through the west. We must be careful to distinguish A_o from Az, which is measured from the north through east.

If the star lies in the western sky, a correction to the auxiliary angle must be made to arrive at the proper value of azimuth. The relationship of the auxiliary angle to the azimuth is as follows: If $0\text{h} \leq H \leq 12\text{h}$, then $Az = A_o$. If $12\text{h} < H < 24\text{h}$, then $Az = 360° - A_o$. Perhaps a more useful way to state this for programmable computers is in the **sine function test**. If $sin\,H \leq 0$, then $Az = A_o$; if $sin\,H > 0$, then $Az = 360° - A_o$.

Consider first the altitude of the given star. Applying the cosine formula to spherical triangle PSZ yields:

$$cos\,(90° - h) = cos\,(90° - \phi)\, cos\,(90° - \delta) + sin\,(90° - \phi)\, sin\,(90° - \delta)\, cos\,H$$

$$sin\ h = sin\ \phi\ sin\ \delta + cos\ \phi\ cos\ \delta\ cos\ H$$

We also can replace H with the difference between local sidereal time (lst), which is to be described shortly, and the right ascension of the object in question, α, where $H = lst - \alpha$.

$$h = sin\ \phi\ sin\ \delta + cos\ \phi\ cos\ \delta\ cos\ (lst - \alpha)$$

Similarly,

$$cos\ (90° - \delta) = cos\ (90° - \phi)\ cos\ (90° - h) + sin\ (90° - \phi)\ sin\ (90° - h)\ cos\ A_O$$

$$sin\ \delta = sin\ \phi\ sin\ h + cos\ \phi\ cos\ h\ cos\ A_O$$

$$cos\ A_o = \frac{sin\ d - sin\ \phi\ sin\ h}{cos\ \phi\ cos\ h}$$

recalling the requirements for deriving azimuth (Az) from the auxiliary angle (A_o).

With the apparent motion of the celestial sphere, the altitude of a celestial object is time dependent. Time dependence enters into the equation for h through the term H, which equals $lst - \alpha$. In the case of azimuth, the time dependence comes through the variable h.

It is interesting to note that the above formulas can be used to direct an altazimuth-mounted "goto" telescope. A command can be used to calculate the elevation and azimuth of a new celestial object once a telescope is aligned with the stars. The servo motors are then driven from its current location through angles equal to changes in azimuth (ΔAz) and elevation (Δh).

We now examine how local sidereal time and the observer's location are integrated into the hour angle term.

5. Sidereal Time and Hour Angle

Figure 18 shows the celestial sphere as seen from a perspective above Earth's north celestial pole, N. Let NM constitute that part of an observer's celestial meridian from the celestial pole down to the celestial equator. Let $N\gamma$ be the equinoctial colure, an hour circle with a right ascension of 0h by definition. Let NS be the hour circle that contains a celestial object of right ascension α.

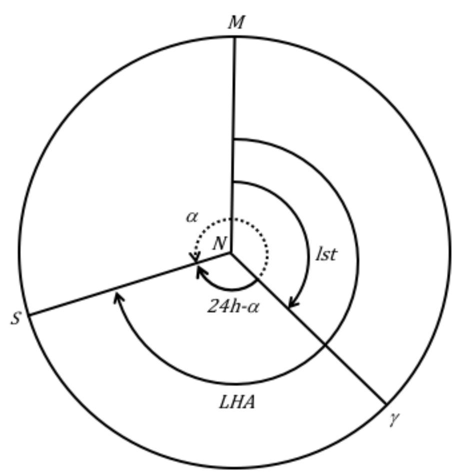


Figure 18. The relationship between α, lst, and LHA.

The **Local Hour Angle** ($\widehat{MN\gamma}$) of the First Point of Aries, γ, constitutes **local sidereal time** that is denoted lst. $\widehat{\gamma MS}$ constitutes the right ascension of the celestial object on NS. $\widehat{\gamma NS}$ is therefore $24h - a$. $\widehat{M\gamma S}$ is the local hour angle, LHA, of the object on NS. Because $\widehat{M\gamma S}$ equals $M\gamma$ plus γS, the relationship between hour angle, sidereal time, and right ascension is as follows:

$LHA = lst + 24h - \alpha$ which is the same as $LHA = lst - \alpha$ due to the fact that $24h$ is the same as $0h$.

The local hour angle of a celestial object and local sidereal time differ from those observed from the prime meridian as a function of both time and observer's longitude. It is possible to redefine local sidereal time as right ascension of an object on an observer's celestial meridian where the hour angle is zero. That is, *the right ascension of a point on an observer's celestial meridian is equal to that observer's local sidereal time.* The relationship between local hour angle, local sidereal time, and the observer's longitude is fundamental to our understanding of the celestial sphere's astronomy.

Sidereal time constitutes a measure of the rotation of the Earth with respect to the celestial sphere. Its basis is the **sidereal day**, a period required for two successive transits of the First Point of Aries. Like right ascension, sidereal time is measured in hours, minutes, and seconds.

The relationship between the hour angle for an observer on the prime meridian, the **Greenwich Hour Angle**, GHA, and the hour angle for the local observer, Local Hour Angle, LHA, of an observer at some non-zero longitude, λ, can be obtained from in inspection of Figure 19. This figure represents a north polar view of planet Earth.

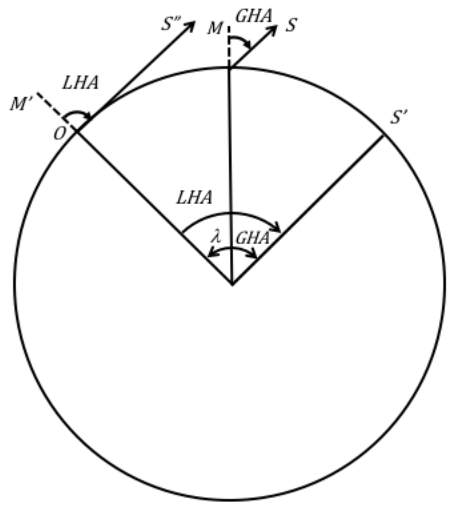


Figure 19. Relationship of the selected angles to selected elements of time.

Let NG represent the prime meridian and GS the direction of an object on the celestial sphere. The extension of NG points to the Greenwich celestial meridian M. $\widehat{MGS}$ is the Greenwich hour angle of the object in question. Construct line NS' parallel to GS. Because $\widehat{MGS}$ is a corresponding angle to GNS', angle GNS' is equal to the GHA of the object at S.

Let O represent the position of an observer, not on the prime meridian whose longitude is λ. Line NO extended to the celestial sphere points to the celestial meridian of O. Construct the line OS" parallel to GS and NS' and extend NO to M', the celestial meridian of the observer. $\widehat{M'OS}$" is the local hour angle of the object in question. Because OS" is parallel to NS', $\widehat{ONS}$' is the corresponding angle of $\widehat{M'OS}$". Hence, $\widehat{ONS}$' equals the local hour angle of object S as seen from O.

Now, $\widehat{GNO}$ represents the longitude of the observer at O. Hence,

$$LHA = GHA + \lambda$$

For this relationship to hold as written, modern longitude sign conventions must be observed. That is, longitude west is taken as negative, whereas longitude east is taken as positive.

Because the hour angle of an object increases with time, its relationship with ordinary clock time must be addressed. Also, because the Earth completes one rotation with respect to the stars (a sidereal day – the basis of star time) in less than 24 hours (a mean solar day – the basis for civil time), the relationships between ordinary clock time and "star time" must be addressed.

5a. Civil Time

Civil time, the ordinary day-to-day time kept with the clocks, watches, cell phones, tablets, and so forth, is a relatively simple concept. It's origin, however, is complex. Civil time is based upon the average diurnal motion of the Earth and the resulting apparent motion of the sun across the sky.

Since time immemorial, humans have kept track of the passage of the daylight hours based upon the sun's motion. The sun rises along the eastern horizon in the morning, is due south at midday, and sets along the western horizon in the evening.

Time once was regulated by reference to this companion through the use of the sundial. People who regulated their lives by the daily motion of the actual sun were making use of **apparent solar time**.

With the advent of uniformly moving clocks, humans began to realize that the sun was an inconstant companion. Its speed across the sky and among the stars was not uniform. The time of midday varied by more than 30 minutes throughout a year.

The sun's irregular motion across the sky is caused by two factors: 1) irregular eastward motion with respect to the background stars, and 2) irregular motion of observers located on the surface of the Earth. The larger and more important variation of apparent solar time is due to the first factor.

Irregularity in the sun's eastward motion among the stars has a seasonal variation due to the obliquity of the ecliptic and the **eccentricity** or non-circular shape of the Earth's orbit.

As the Earth nears the sun, the rate of its orbital motion increases. This causes a corresponding change in the rate of the sun's motion along the ecliptic. As the Earth pulls away from the sun, the Earth's orbital motion slows, and the sun's rate of motion along the ecliptic decreases.

Because the ecliptic is inclined to the celestial equator by about $23.44°$, the component of the sun's eastward motion among the stars increases and decreases. When the sun is near a solstice position on the ecliptic, most if not all of its motion is directed toward the east. When the sun nears an equinox position, the sun has a significant component of motion directed either north or south.

The time at any one spot based upon the average motion of the sun is called **mean solar time** or, more commonly, **local mean time**. Local mean time is based on the 24-hour **mean solar day**. This is the average time required for the **fictitious mean sun** to cross the observer's meridian twice. The fictitious mean sun is an imaginary body that moves at a uniform rate eastward along the celestial equator.

The difference between apparent solar time and mean solar time can amount to as much as 16 minutes at certain times of the year. This difference between these two times, apparent solar time minus mean solar time, is known as the **equation of time**.

With the advent of rapid communications and travel during the mid 19th century, it became evident that the only way to regulate travel and clocks was by setting up time zones. At the instigation of the American railroads, time would no longer be regulated by the sun's apparent position; neither would each town keep its own time based on the sun's motion. Instead, timekeeping would be based upon **time zones**, regions roughly $15°$ of longitude wide, adapted to political boundaries. Hypothetically speaking, the basis for each time zone is a **central meridian**, one of 24 meridians of longitude that were integer multiples of $15°$**.** Ideally, each time zone would stretch $7.5°$ of longitude east and west of the central meridian, and everyone within that time zone would observe the same time. Today civil time and time zones are regulated by the Department of Commerce in the United States.

North America would have four such time zones (Eastern, Central, Mountain, and Pacific) and four standard meridians ($75°$, $90°$, $105°$, and $120°$ west longitude respectively). The inhabitants of each zone would keep **standard time**, the mean solar time kept by a clock located on the central meridian of the time zone. Time would vary by one hour from one time zone to the next. Moving westward from one time zone into the next, an observer would simply set his clock back one hour and vice versa.

There are "problems" associated with the artificial construct of time zones. Persons on the eastern side of the time zone might have sunrise at 5:30 AM, whereas persons living on the western side of the time zone might have sunrise at 6:30 AM. This has caused some fragmentation of certain time zone boundaries. Also, a source of boundary problems is large population centers. Gary, Indiana, for instance, wants to be in the same time zone as Chicago, its larger neighbor. As a result, the tip of northwest Indiana is not in the same time zone as the rest of Indiana. Additionally, the remainder of Indiana has chosen to remain always on standard time (like the states of Arizona and Hawaii), whereas some states are now planning to move permanently to daylight saving time such as Illinois.

If events such as rising and setting are calculated for the standard meridian of the time zone, then a **longitude time correction**, ltc, must be applied to observers at the various locations within the time zone at the same latitude. For every degree farther east in longitude, one was in the time zone, events in the heavens occur four minutes earlier that at the standard meridian. For every degree west, celestial events occur 4 minutes later than at the standard meridian. The correction is negative if the observer is east of the time zone's standard meridian and positive if west.

Local mean time can be related to standard mean time in the following way:

$$\text{local mean time} = \text{standard mean time} + \text{longitude time correction}$$

With advances in communication and the attendant apparent shrinkage of the global village, the 24 time zones around the globe also have limitations. **Universal Time** (UT) was initially established in the late 1800s to serve as an international reference for the determination of longitude using celestial navigation. Today UT is regularly used by astronomers and others (e.g., airline, ship, and train companies) with a need to avoid confusion between time zones. Universal Time is essentially the hour angle of the fictitious mean sun observed at the prime meridian plus 12 hours. It is expressed in the 24-hour system whereby 1:00 PM is 13:00 and so on.

Universal Time is a measure of time that conforms very closely to the daily motion of the fictitious mean sun. Universal Time is formally defined by an equation that relates the average rotation rate of the Earth and the daily motion of the fictitious mean sun. It is determined from observations of the stars. Universal Time so defined is termed UT0. UT0 increases at a non-uniform rate due to variations in the rotation rate of the Earth.

If small corrections are applied for the shifts in longitude of the observing site due to **polar motion**, the irregularly varying motion of the terrestrial poles with respect to the Earth's crust, then UT0 becomes UT1.

UT1 is still somewhat variable due to seasonal changes in the rotation rate of the Earth. These variations have been empirically determined and can be applied as corrections to UT1 to produce a much more uniformly flowing system of time measure known as UT2.

UT2 is nearly uniform time. High precision atomic clocks keep time that is still more uniform, however. Atomic clocks measure the passage of time about a million times more accurately than can be determined through astronomical observations. Because of these differences yet another time convention was established, which is called **Coordinated Universal Time** or simply UTC. UTC serves as the basis of all civil timekeeping.

Because Earth's overall rotation rate is decreasing due to tidal effects of the sun and moon, atmospheric loading, plate motions, and a variety of other physical phenomena, the day is not precisely $86{,}400$ seconds in duration, but slightly longer. Because of the increasing length of the day, a **step adjustment**, essentially a leap second, is introduced into UTC whenever the difference UT1 - UTC exceeds 0.7 seconds. On the average, a leap second is added or subtracted every 1 to 3 years and usually at the end of either June 30 or December 31.

International Atomic Time, TAI, is pure atomic time that is the basis of UTC. TAI is never adjusted and differs from UTC only as a consequence of the addition of leap seconds. The difference TAI - UTC had increased to 37 seconds as of January 1 2017, when another leap second was added.

Yet another measure of time, **Terrestrial (Dynamical) Time** (TDT = deprecated abbreviation or TT = preferred abbreviation), increases at the same rate as TAI, but TT runs ahead of TAI by precisely $32.184s$ for historical reasons. TT is the time normally used to define motions of bodies within the solar system. As of the start of 2020, TDT-UTC = ΔT = $38.568s + 32.184s$ = $70.752s$.

The relationship between Coordinated Universal Time (UTC) and standard time for each of the American time zones is as follows:

Eastern Standard Time + 5h = UTC
Central Standard Time + 6h = UTC
Mountain Standard Time + 7h = UTC
Pacific Standard Time + 8h = UTC
Alaska Standard Time + 9h = UTC
Hawaii Standard Time + 10h = UTC

If daylight saving time is in effect, then the number of hours added according to the above list is decreased by 1h (e.g., CDT + 5h = UTC).

Coordinated Universal Time can be obtained from short wave signals broadcast by station WWV of Fort Collins, Colorado. Continuous time signals can be heard by tuning receivers to 2.5, 5, 10, 15, or 20 megahertz or via CHU in Canada by tuning to 3.330, 7.335, or 14.670 megahertz. Signal strength and clarity are dependent upon atmospheric conditions and time of day.

During certain times of the year, clocks are set forward one hour to establish **daylight saving time** (DST) under the regulation of the U.S. Department of Commerce and subject to states' rights. (Arizona, Hawaii, and parts of Indiana do not observe Daylight Saving Time; other states appear to be moving toward permanent Daylight-Saving Time.) The effect of DST is to "add" one additional hour to the evening hours by "taking" it from the morning hours.

If, following these time zone corrections, you arrive at a time greater than 24 hours, subtract 24 hours and add one day to the date. For instance, the Coordinated Universal Time corresponding to 8:00 PM Central Daylight Time on June 8 is 25 hours UTC June 8 or 1H UTC June 9.

The relationship between local mean time, UTC, and the observer's longitude, λ, is as follows:

$$\text{local mean time} = \text{UTC} + \lambda$$

keeping in mind that west longitudes are considered negative and expressed in time equivalence based upon the conversion factor of one hour per 15°.

5b. Sidereal Time

Earth completes $360°$ of rotation with respect to the stars every sidereal day. However, the length of the sidereal day varies because the rate at which the Earth rotates is not constant. Variations in the Earth's spin rate are due to lunar and solar tidal effects, core-mantle interactions, and seasonal meteorological changes. Also, shifts in the position of the Earth's crust with respect to the Earth's axis of rotation induce small changes in the apparent length of the sidereal day.

Because these changes and their magnitudes are generally unpredictable, astronomers have chosen to deal with **mean sidereal time**. Mean sidereal time is the hour angle of the **mean equinox of date**, necessarily the current position of the First Point of Aries, the location of which is affected by the phenomenon of **precession**. Precession is the continual reorientation of the Earth's axis in space as a result of Earth-Moon-Sun gravitational interactions. Mean sidereal time is set to increase at a rate that tends to average out the phenomena that cause variations in the length of the actual sidereal day.

On average, a star returns to the same place in the sky roughly every 23 hours, 56 minutes, and 4.1 seconds. This time constitutes the **mean sidereal day**. Because of the sun's ecliptic motion and precession, the period required to bring the sun back to the same place in the sky is longer than the time for the Earth to complete one rotation with respect to the stars.

As the Earth orbits the sun in about 365.242 days (a seasonal or tropical year), the sun moves $360°$ along the ecliptic. Daily, the sun appears to move an average of $0.9856°$ per day $(360°/365.242d)$ eastward along the ecliptic. As a result, the Earth has to turn through this extra distance to bring the sun back to the same point in the sky. This requires approximately 3 minutes, 55.9 seconds, which yields the 24-hour time period we call the mean solar day. Because 24 hours of mean sidereal time pass in $23h$, 56m, and 4.1s of civil time, sidereal time increase at a rate that is currently 1.0027283 times faster than that of civil time.

The **local mean sidereal time** ($LMST$) differs from **Greenwich mean sidereal time** ($GMST$) at a given instant of time as a function only of longitude, λ, in the following fashion:

$$LMST = GMST + \lambda$$

where λ is expressed in time-equivalent units and taken as negative if west and positive if east.

Using this relationship, the local hour angle of a celestial object can be readily calculated, given its right ascension and local mean sidereal time. Local mean sidereal time can be readily calculated for any instant in civil time.

5c. Calculating Local Mean Sidereal Time

As noted above, $LMST$ can be found simply from a knowledge of $GMST$ and the time equivalence of λ. Coming up with $GMST$, however, is complicated. $GMST$ at a particular time on a particular date slowly changes because of irregularities in the civil calendar, precession, and nutation. Table 1 lists the $GMST$ for 0h UTC on January 1 for several years. From this table, it should be clear that the determination of $GMST$ is not a trivial undertaking.

Year	**$GMST$ (0h UTC)**	**Year**	**$GMST$ (0h UTC)**
2021	6h 43m 28s	2026	6h 42m 39s
2022	6h 42m 30s	2027	6h 41m 42s
2023	6h 41m 33s	2028	6h 40m 45s
2024	6h 40m 36s	2029	6h 43m 44s
2025	6h 43m 36s	2030	6h 42m 47s

Table 1. Greenwich Mean Sidereal Times for 0h UTC for January 1 for several year, 2021-2030.

The process of determining the value of $GMST$ is complex and beyond the scope of this work. However, an example is provided to show how to calculate $LMST$ for completeness. Consider the following example.

An observer located at longitude $88.97408°$ west (time equivalent equals $5h\ 55m\ 54s$) wishes to find local mean sidereal time at 9:00 PM CDT on May 17, 1992. The observer is located in the Central Time Zone that is observing Daylight Saving Time.

Convert the observer's AM/PM time to the 24-hour system by adding 12 hours due to it being PM:

9 PM CDT (5/17) + 12h = 21h CDT (5/17)

To convert this 24-hour system value of CST to UTC, add 5 hours because DST is in effect (add 6 hours if DST is not being observed).

21h CST (5/17) + 5h = 26h UTC (5/17)

Now, 26h UTC on May 17 is the same thing as 2h UTC on the 18th. That is,

26h UTC (5/17) = 2h UTC (5/18)

Using a United States Naval Observatory *Astronomical Almanac* or similar, the observer determines that the GMST for 0h UTC is:

$GMST$ (0h UTC) = 15h 43m 43s

Two hours of civil time later, sidereal time will have increased by 1.0027378 x 2h or 2h 0m 20s. Hence,

$GMST$ (2h UTC) = 15h 43m 43s + 2h 0m 20s = 17h 44m 3s

Because LMST = GMST + λ, we have after converting the observer's west longitude to time equivalent (recall that –88.97408° = –5h 55m 54s):

$LMST$ = 17h 44m 3s – 5h 55m 54s = 11h 48m 09s

Now that we have a handle on how to find local mean sidereal time, we can put this information and our knowledge of spherical astronomy to good use. In the following section, the reader learns how to calculate a simple **nomogram**, a pictorial representation of the heavens. earns

6. Further Applications of Spherical Astronomy

What follows are several examples of the application of spherical astronomy to practical problems. These examples deal with the angle of inclination of the ecliptic to the horizon as a function of sidereal time, archaeoastronomy alignments, rising and setting azimuths, and the preparation of two **nomograms** or astronomical charts.

EXAMPLE 6: Angle Between Ecliptic and Horizon

It has been said that for a given elongation of a planet, the planet may appear either high up in the sky or low down depending on the angle that the ecliptic makes with the horizon. For instance, it is a well-known fact that Mercury can be viewed best during autumn evenings and spring mornings. This is so because the angle that the ecliptic makes relative to the horizon is greatest under these circumstances. What angle does the ecliptic make with the horizon as a function of local sidereal time? The answer can be derived from a study of Figure 20.

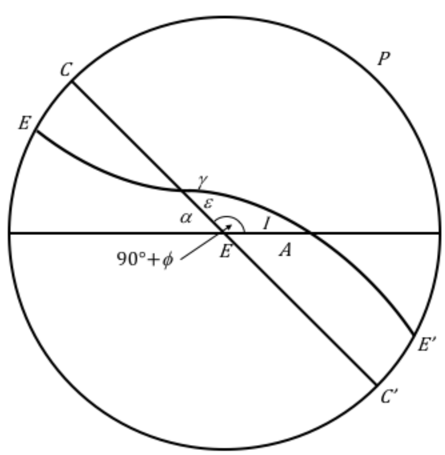


Figure 20. Inclination of the ecliptic relative to the horizon as a function of hour angle.

In this figure, CC' represents the celestial equator that intersects the horizon at E. From our initial results in section 2a above, we know that the angle at E equals $90° - \phi$. Its supplementary angle, E' equals $90° + \phi$. If γ represents the First Point of Aries, the distance γE equals the right ascension of an object at its rising. Because the ecliptic intersects the celestial equator at angle ε, we can now solve for I, the inclination of the ecliptic relative to the horizon. Applying the polar version of the cosine formula

$$- \cos A = \cos B \cos C - \sin B \sin C \cos a$$

we get:

$$\cos I = -\cos (90° + \phi) \cos \varepsilon + \sin (90° + \phi) \sin \varepsilon \cos \alpha$$

And because $\cos (90° + \phi) = - \sin \phi$ and $\sin (90° + \phi) = \cos \phi$

$$\cos I = \sin \phi \ \cos \varepsilon + \cos \phi \sin \varepsilon \cos \alpha$$

Because by definition $H = lst - \lambda$ and the hour angle of an object rising due east has been shown via Example 3 above to be 90°, α can be rewritten as $lst - 90°$ or

$$\cos I = \sin \phi \ \cos \varepsilon + \cos \phi \sin \varepsilon \sin lst$$

Hence, when lst equals $90°$ $(6h)$ or $270°$ $(18h)$, the value of $\sin lst$ is $+1$ and -1, respectively, and I is at extreme values. For $40°$ north latitude and $\varepsilon = 23.44°$, I reaches a maximum value of $73.4°$ and a minimum value of $26.6°$. The maximum values of I occur at the times of autumn equinox sunrise and spring equinox sunset (the best times to view Mercury). The minimum values of I occur at the times of the spring equinox sunrise and autumn equinox sunsets (the worst times to view Mercury). The above relationship reduces to two straightforward maximum and minimum formulas for I when the First Point of Aries is on the horizon, either east or west:

$$I_{max} = 90° - \phi + \varepsilon$$

$$I_{min} = 90° - \phi - \varepsilon$$

Because inferior planets near greatest eastern elongation are viewed following sunset, the absolute best times to observe them would be during late winter sunsets when the sun would be below the horizon at $6h$ sidereal time. This occurs in the days just before the spring equinox. Similarly, the absolute best time to view inferior planets at greatest western elongation would be when the sun would be just below the horizon at $18h$ sidereal time, just after the autumn equinox. At these times, the planet would have its greatest altitude above the horizon for a given elongation with the sun still below the horizon.

EXAMPLE 7: Archaeoastronomy Alignments

An archaeological site with suspected astronomical alignments is found at $32°$ north latitude. An analysis of several possible lines of sight yields one that aligns with the horizon giving a setting azimuth of 297° ($A_o = 63°$) at an elevation of 2°. To what celestial event – if any – does this alignment correspond? The situation with respect to the celestial sphere is shown in Figure 21.

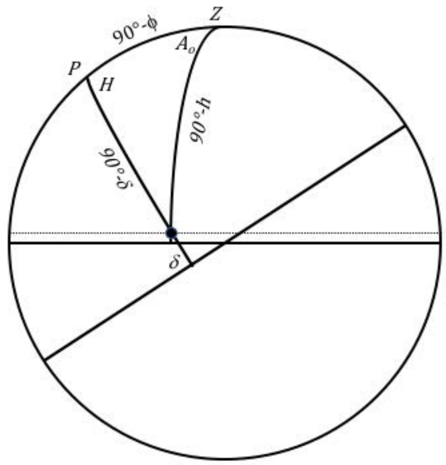


Figure 21. Finding the declination of the setting sun given a particular setting azimuth.

From an analysis of the spherical triangle in Figure 20 containing the zenith, celestial pole, and point on the horizon, we arrive at the following statements using the law of cosines:

$$\cos(90° - h) = \cos(90° - \phi)\cos(90° - \delta) + sin(90° - \phi)\sin(90° - \delta)\cos A_o$$

$$sin\,\delta = sin\,\phi\; sin\,h + cos\,\phi\; cos\,h\; cos\,A_o$$

Inserting the values for ϕ, h, and A_o yields a value for declination equal to $+23.8°$, a nice match for the sun's declination on the summer solstice ($+23.44°$). Hence, this well could be a summer solstice marker.

This analysis does not take into account the refraction due to the Earth's atmosphere. That should not be a reason for concern, however, because many astronomical alignments are rough at best.

Because the obliquity of the ecliptic changes very little over hundreds of years, searching for solar and even lunar alignments are easily accomplished. If one wants to establish stellar alignments, then the effects due to precession of Earth's axis must be taken into account.

EXAMPLE 8: Rising and Setting Azimuths

The rising and setting azimuths may be obtained from the expression found in Example 7. Solving the equation for A_o yields:

$$\cos A_o = \frac{sin\,\delta - sin\,\phi\, sin\,h}{cos\,\phi\, cos\,h}$$

If the altitude h is replaced by $90° - z$, the relationship becomes:

$$cos\,A_o = \frac{sin\,\delta - \sin\phi\, cos\,z}{cos\,\phi\, sin\,z}$$

Because the inverse-cosine function returns values only in the range of $0°$ to $180°$, how then does one arrive at setting azimuths that would range between $180°$ and $360°$? Setting azimuths will are equal to $360°$ minus the rising azimuth (assuming the same declination, latitude, and a level horizon with $h = 0°$) and can be readily found via the following expression:

$$Az_{setting} = 360° - Az_{rising}$$

If one is carefully calculating rising and setting azimuths of sun, moon, stars, or planets, then the effects due to refraction by the atmosphere must be taken into account. Additionally, if one is calculating such azimuths for the sun and moon, apparent semidiameters and refraction also must be taken into account. The moon's position must also be corrected for parallax. The methods for managing these considerations are covered in later sections of this article.

EXAMPLE 9: Astronomical Nomogram #1

An observer is located at $88.974°$ west longitude, $40.502°$ north latitude. He wants to prepare a diagram for his local newspaper that shows the relative positions of Uranus, Neptune, the constellation Sagittarius, and the horizon at 11:30 PM on July 20, 1992. To do so, the observer is required to calculate the altitudes and azimuths of each celestial object. From prior findings we have in the order of use the following equations:

$$H = lst - \alpha$$

$$sin\, h = sin\, \phi\, sin\, \delta + cos\, \phi\, cos\, \delta\, cos\, H$$

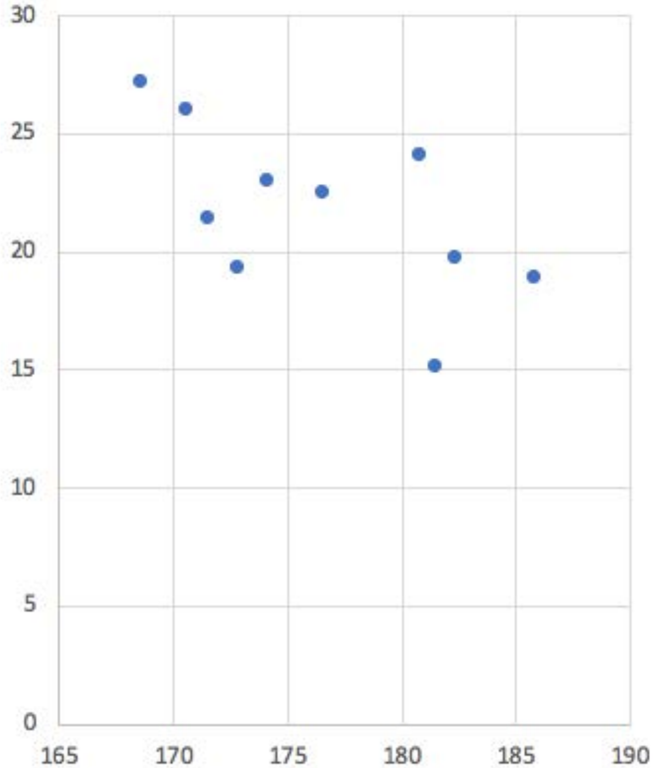
$$cos\, A_o = \frac{sin\, \delta - sin\, \phi\, sin\, h}{cos\, \phi\, cos\, h}$$

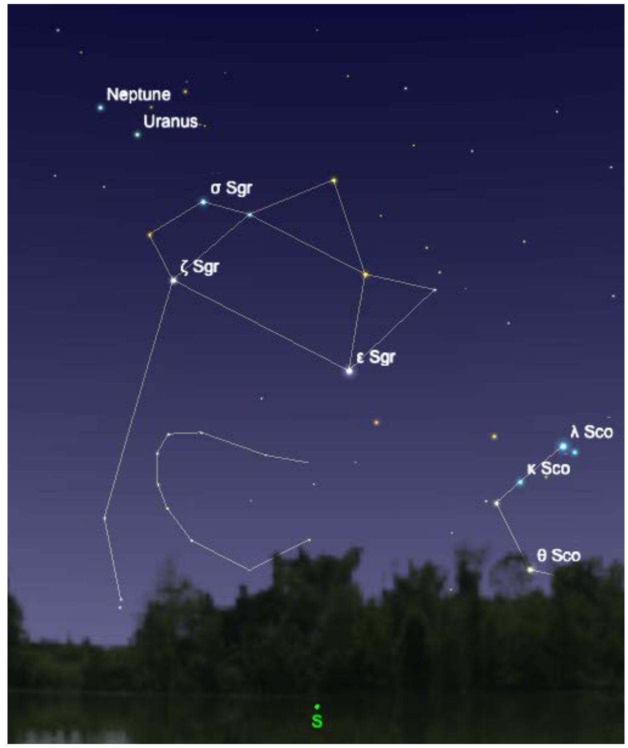
Our amateur astronomer begins the calculations by determining the local mean sidereal time for 11:30 PM and arrives at $18h\ 30m\ 53s$ or, for purposes of calculation, $277.721°$. Using a table of right ascensions (epoch 1992.6, converted to decimal degrees) and declinations for each star, Uranus, and Neptune, the observer, calculates the data shown in Table 2. (Corrections for refraction not applied.)

Object	$\boldsymbol{\alpha}(°)$	$\boldsymbol{\delta}(°)$	$\boldsymbol{H}(°)$	$\boldsymbol{h}(°)$	$\boldsymbol{Az}(°)$
γ SGR	271.338	-30.425	$+6.384$	18.875	185.814
δ SGR	275.125	-29.817	$+2.596$	19.672	182.383
ε SGR	275.925	-34.383	$+1.796$	15.151	181.536
λ SGR	276.875	-25.433	$+0.846$	24.104	180.830
ϕ SGR	281.300	-27.000	-3.579	22.458	176.548
σ SGR	283.700	-26.300	-5.979	23.000	174.174
ζ SGR	285.525	-29.530	-7.804	19.283	172.825
τ SGR	286.625	-27.683	-8.904	21.359	171.541
Uranus	286.850	-22.967	-9.129	25.993	170.630
Neptune	288.600	-21.583	-10.879	27.124	168.632

Table 2. Hand-calculator-based elevations and azimuths of stars and planets for making Figure 22a.

Our amateur astronomer has been very careful to determine which side of the meridian the objects appear by applying the sine function test before finding azimuth from the auxiliary angle. Plotting the tabulated values of h and Az on a rectangular grid yields results shown in Figure 22a.





Figures 22a and 22b. Nomogram shows Uranus, Neptune, and the stars of Sagittarius at 11:30 PM on July 20, 1992, from Central Illinois. Figure 22a created using MS *Excel*. Figure 22b from *SkySafari 6 Plus*.

Of course, modern computer programs and apps for cell phones and tablets can perform these calculations and produce beautiful renderings in a flash. Figure 22b shows one such chart produced with *SkySafari 6 Plus* (version 6.5.0, Simulation Curriculum Corp., copyright 2014-2018) on a desktop computer. Again, the idea behind this TCAA Guide is not to have you do such calculations by hand, but to understand how they are done. If you are a programmer, then you can prepare the programs necessary to make such calculations.

EXAMPLE 10. Solar Right Ascension and Declination based on ecliptic longitude

Another useful result from spherical astronomy can be derived from Figure 23. This figure shows the sun at a point along the ecliptic. Its position there is shown in relationship to the celestial equator and the First Point of Aries (γ). The distance of the sun from γ measured along the ecliptic is known as ecliptic longitude represented with the symbol λ. The distance north or south of the celestial equator is declination (δ) and eastward from the First Point of Aries measured along the celestial equator right ascension (α).

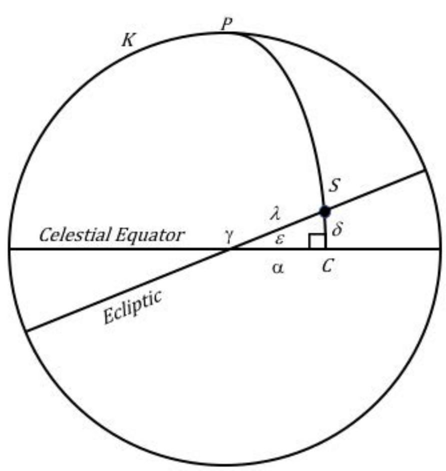


Figure 23. Solar right ascension and declination based on ecliptic longitude.

Recall the four-parts formula:

$$cos\ (inner\ side)\ cos\ (inner\ angle) = sin\ (inner\ side)\ cot\ (other\ side) - sin\ (inner\ angle)\ cot\ (other\ angle)$$

From Figure 22, we derive the following relationship using the four-parts formula:

$$cos\ \alpha\ cos\ 90° = sin\ \alpha\ \cot\delta - sin\ 90°\ cot\ \varepsilon$$

which reduces to

$$sin\ \alpha\ \cot\delta = cot\ \varepsilon$$

or

$$\tan\delta = sin\ \alpha\ tan\ \varepsilon$$

Given this relationship, the declination of the sun can be found for any right ascension with only knowledge of the value of the obliquity of the ecliptic, ε, which is roughly equal to $23.44°$. For instance, when the right ascension of the sun is $6h$ $(90°)$, then $\delta = \varepsilon$. When $\alpha = 18h$ $(270°)$, then $\delta = -\varepsilon$ and so forth.

EXAMPLE 11: Astronomical Nomogram #2

A more sophisticated nomogram can be prepared that exhibits the location of a celestial object as a function of the sun's zenith distance. This is especially useful when preparing charts showing planets in the evening or morning twilight sky. Charts for the evening can be prepared for the end of **civil twilight** (sun's zenith distance, z, equals 96°), **mid twilight** ($z = 99°$), the end of **nautical twilight** ($z = 102°$), the end of **astronomical twilight** ($z = 108°$), or at any point in between.

The standard formulas for the calculations of altitude and azimuth are given in Example 9. The dependence on local sidereal time can be eliminated through appropriate substitutions. Let $H_\odot$ and H_p be the hour angles of the sun and planet, respectively. Let $\alpha_\odot$ and α_p be the right ascensions of the sun and planet, respectively. Note first that we have the hour angles of the sun and planet respectively:

$$H_\odot = lst - \alpha_\odot$$

and

$$H_p = lst - \alpha_p$$

where lst is the local sidereal time at chart time. Solving each of the above equations for lst, equating, and solving for H_p yields:

$$H_p = H_\odot + \alpha_\odot - \alpha_p$$

Given a solar zenith distance of, say, 99° (the sun's center is 9° below the horizon – mid twilight), the sun's hour angle is readily determined from the equation $H_\odot = lst - \alpha_\odot$. With the values of $\alpha_\odot$ and α_p known, it is easy to find H_p. With the right ascension of the sun known, the declination of the sun, $\delta_\odot$, can readily be found using the equation of Example 10. It is then easy to calculate the altitude and azimuth of the planet using the formulas of Example 9. Figure 24 shows one instance of such a calculation. Here, the position of Venus is shown in the western evening sky at 30-day intervals at the end of civil twilight (sun's center 6° below the horizon).

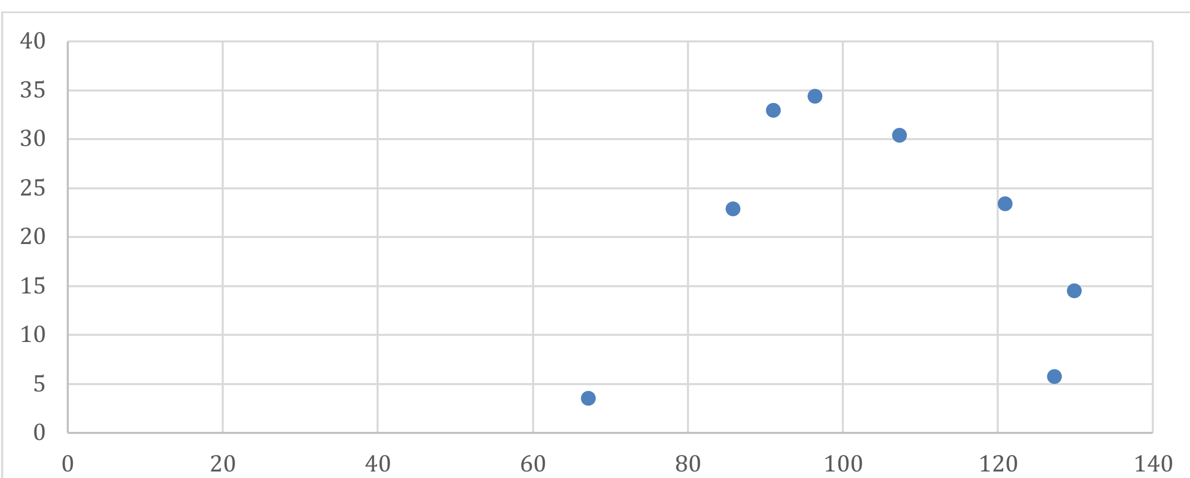


Figure 24. The position of Venus in the eastern predawn sky with the sun 9° below the horizon at 30-day intervals, June 15, 2020, through January 11, 2021.

Such nomograms, as shown in Figure 24, can be generated with the use of computer programs and apps for smartphones and tablets. Unfortunately, none of them at this time directly produced such charts. Charts for various dates must be generated separately, and the images stacked, as shown in Figure 25.

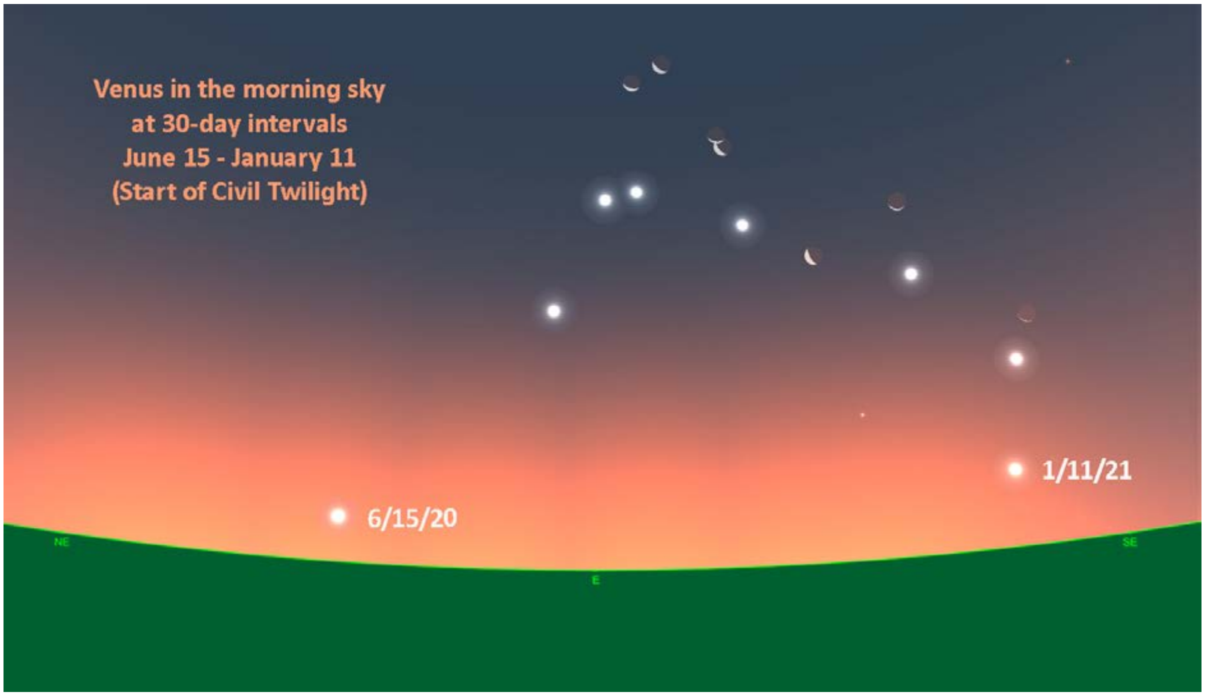


Figure 25. Venus at 30-day intervals at the start of civil twilight. This image is a stack of 8 individual images produced using *SkySafari 6 Pro* and *Photoshop*. Note also the appearance of Mercury and the moon.
EXAMPLE 12: Times of Sunrise and Sunset

It would seem that calculating the times of sunrise and sunset would be a rather straight forward task. Surprisingly, this is not the case. The calculations are attended by a considerable number of complicating factors. Among them are the following:

- **Uncertainty of the sun's position.** Calculations of the times of sunrise and sunset depend upon a knowledge of the sun's position at those times and, consequently, the associated hour angles. Unfortunately, we can't know the sun's position with precision until we know the times of sunrise and sunset. The best we can do is to estimate the time of these phenomena and then determine a new value for the sun's right ascension and declination and then recalculate the hour angles. This becomes an iterative process, successively reducing error in the sun's position along with hour angles.
- **Irregular ecliptic motion of sun.** Making the situation more complicated is the fact that the sun's right ascension is affected by its eastward motion along the ecliptic, which is itself irregular. The sun's rate of motion along the ecliptic is at a maximum when Earth is nearest the sun and, at a minimum when Earth is farthest from the sun. The difference between right ascension of the mean sun and its actual right ascension is exacerbated by the fact that the ecliptic is inclined to the celestial equator (along which right ascension is measured) by the obliquity of the ecliptic, about $23.44°$.
- **Sun's change in right ascension.** The sun's change in right ascension during the morning hours delays its time of meridian transit; similarly, the sun's motion during the afternoon hours delays its setting time. These delays typically range from 1 to 2 minutes each depending on the time of year and the observer's latitude.
- **Time of sun's meridian transit.** Variations in the sun's motion along the ecliptic can be reduced to a single formula, the so-called equation of time, E. Here E is the difference between the right ascension of the apparent (actual) sun (RAS) that moves along the ecliptic at variable speed and the right ascension of the fictitious mean sun (RAMS) that moves along the celestial equator at a steady pace. The equation of time takes on values ranging from $-14m\ 15s$ to $+16m\ 25s$, as shown in Table 3.

Aspect	Value	Date
minimum	$-14m\ 15s$	February 11
zero	$0m\ 0s$	April 15
maximum	$+3m\ 41s$	May 14
zero	$0m\ 0s$	June 13
minimum	$-6m\ 30s$	July 26
zero	$0m\ 0s$	September 1
maximum	$+16m\ 25s$	November 3
zero	$0m\ 0s$	December 25

Table 3. Values of E, the equation of time, for various points throughout the year. $E = RAS - RAMS$. (Epoch 2000)

The equation for E deals with aspects of celestial mechanics that are not appropriate to this short treatise. Ergo, this topic is not addressed here. Those seeking details and formulae will are referred to Jean Meeus' *Astronomical Algorithms* (2nd edition, 1999, Willmann-Bell, Inc.)

- **Atmospheric refraction.** The variance of atmospheric temperature, temperature gradients associated with frontal systems, relative humidity, and pressure all induce changes in atmospheric refraction. Atmospheric refraction at the horizon is about $35.4'$ for a temperature of $10°C$ and an atmospheric pressure of $1013.25hPa$ in the visible part of the spectrum. A value of 35' is adopted here for calculations.
- **Sun's apparent diameter.** The sun's apparent diameter as seen from Earth varies, ranging from $31'\ 27''$ at the time of aphelion to $32'\ 32''$ at the time of perihelion. An average value of 32' is therefore adopted for the sake of convenience. The semidiameter value of 16' is used in calculations, defining sunrise and sunset as those points in time when the sun's upper limb is on the horizon.
- **Irregular horizon.** The observer's horizon does not typically everywhere have a zenith distance of $90°$. Most horizons are undulating, with the zenith distance varying due to undulations in the landscape. The horizon is not uniform except at sea, and even its zenith distance can be affected by the observer's height above the water's surface.
- **Observer height.** The height of the observer above the surface of Earth can produce a "dip" of the horizon resulting in earlier sunrises and later sunsets.
- **Longitude time correction.** Because civil time within a time zone is standardized, everyone within a time zone observes the time kept on the time zone's central meridian. Corrections dealing with the times of sunrise and sunset must be made to account for any longitude difference. Recall that Earth turns through 15°/1hr. Ergo, if a person is 1° *east* of the standard meridian of a time zone, events occur there 4 minutes earlier (60m/15° = 4m/1°) than they do if on the time zone's central meridian. Similarly, if a person is 1° *west* of the central meridian of a time zone, events occur there 4 minutes later than if they are on the central meridian. This difference in times is known as the longitude correction and is denoted here by $\Delta\lambda$.
- **Daylight Saving Time.** When Daylight Saving Time (DST) is in effect, $1h$ is added to the standard time in a time zone.

All of these factors influence the times of sunrise and sunset. As a result, calculations of the times of sunrise and sunset can be several minutes off from the actual events. When combined, these factors can easily cause differences between actual and predicted times of up to ±2 minutes, even for a constant horizon elevation of $0°$. Therefore, using approximations for the times of sunrise and sunset generally suffices.

Now, the sun's hour angle at rising and setting can be used to determine the times of sunrise and sunset. From Example 4, we have the following relationship for the hour angle of solar risings and settings that includes consideration of atmospheric refraction and the sun's semidiameter:

$$H = \frac{1}{15} cos^{-1}((cos\, 90.85° - sin\ \phi\ sin\ \delta)/(cos\ \phi\ cos\ \delta))$$

Recall that the hour angle of rising is taken as negative and the hour angle of setting if taken as positive.

Using the sun's mean declination for the day (δ at the time of meridian transit), the hour angles or rising and setting can be found. The first approximations for the times of sunrise and sunsets are, therefore, as follows (keeping mind that $H_{rise} < 0$).

$$T_{rise} = T_{transit} + H_{rise}$$

$$T_{set} = T_{transit} + H_{set}$$

Now, if the sun's center crossed the meridian at 12:00 PM each day, calculating the times of sunrise and sunset would be simply a matter of adding H_{rise} to 12 to get the time of sunrise or adding H_{set} to 12 to get the time of sunset. Unfortunately, it does not work well for a variety of reasons, as noted at the start of this section. For instance, we need to make corrections for the equation of time, longitude correction for the observer's position within the time zone, as well as Daylight Saving Time (if appropriate).

This article is not designed to make the reader an expert calculator of such phenomena but, instead, to help the reader understand how calculations such as the times of sunrise and sunset are made. A general example should suffice. Those seeking more technical information should turn to publications such as Jean Meeus' *Astronomical Algorithms* (2nd edition, 1999, Willmann-Bell, Inc.)

A fair approximation for the *standard* time of the sun's meridian transit for an observer not on the central meridian of his or her time zone can be written as follows:

$$T_{transit} = 12h \;+\; \Delta\lambda - E$$

Again, if Daylight Saving Time is in effect, then $1h$ must be added to $T_{transit}$.

By way of example, let's determine the time of the sun's meridian transit on May 20, 2020, for a hypothetical observer located at $88.9471°$ west longitude, which is about 1° east of the time zone's standard meridian. The longitude correction, $\Delta\lambda$, is $-4m\ 6s$. The equation of time, E, on this date, is $+3m\ 24s$. DST is in effect, so *1h* must be added to the calculated times. Hence,

$$T_{transit} = 12h - 4m\ 6s - 3m\ 24s + 1h = 12h\ 52m\ 30s$$

Now, the hour angles of rising (taken as negative) and setting (taken as positive) may be added to $T_{transit}$ to arrive at the times of sunrise and set:

$$H_{rise} = H_{set} = \pm 7h\ 18m\ s_{\text{able}}$$

$$T_{rise} = T_{transit} + H_{rise}$$

$$T_{rise} = 12h\ 52m\ 30s - 7h\ 17m\ 55s = 5h\ 34m\ 34s \;=\; 5{:}\,34{:}\,34\ AM$$

$$T_{set} = T_{transit} + H_{set}$$

$$T_{set} = 12h\ 52m\ 30s + 7h\ 17m\ 55s = 8h\ 10m\ 54s \;=\; 8{:}\,10{:}\,54\ PM$$

7. A Word about Twilight

If you are to make nomograms such as those above, it is helpful to know what the sky looks like at the beginnings and ends of civil, mid, nautical, and astronomical twilight. Table 4 gives some idea of what to expect in terms of sky appearances at these times for $40°$ north and southern latitudes. Twilights are most prolonged at the summer solstice and shortest at the winter solstice. Nearer the equator durations of twilight are shorter and nearer the poles durations of twilight are longer.

End of evening/start of morning:	$h_\odot$	Time from sunrise/set	Atmospheric Appearance	Stellar Appearance
civil twilight	$-6°$	Ranges from 31m (Dec.) to 34m (June).	The zenith is light pale blue while the direction opposite the sun's location is dimmer. Area above sun blue with yellow lower down.	Brighter planets and 1st magnitude stars are readily visible higher up in the sky, less so close to sun. Other stars not yet visible.
mid twilight	$-9°$	Ranges from 49m (Dec.) to 55m (June).	There is a light pale blue glow in the sky above the sun. The zenith is ashen blue.	Only 1st and 2nd magnitude stars visible along with all planets other than Uranus and Neptune. Brighter star patterns higher up in the sky are visible. Brighter planets near sun are readily visible.
nautical twilight	$-12°$	Ranges from 65m (Dec.) to 76m	There is a faint ashen glow in the sky above the sun. The zenith and	1st through 4th magnitude stars and all planets other than Uranus

		(June).	that part of the sky opposite the sun are fairly dark.	and Neptune are visible from a dark rural setting. Urban settings are about as dark as they can be to get due to light pollution.
astronomical twilight	$-18°$	Ranges from 98m (Dec.) to 126m (June).	There is no light in the sky coming directly from the sun before the start or after the end of astronomical twilight. The sky is as dark as it can appear.	Entirely dependent upon ambient conditions. In a dark rural setting, the limiting magnitude might be 6.0. or higher. In an urban setting, only the brightest stars might be visible.

Table 4. Atmospheric and stellar appearances in a clear sky at different points during twilight.

8. The Effect of Parallax on the Moon's Altitude

Thus far, in our preparation of the nomogram, we have not touched upon the moon. Because the moon is not at a great distance from the Earth, the position of an observer (offset from the center of the Earth) can affect the apparent position of the moon in the sky. This apparent shift in the position of the moon due to the observer's position is called **lunar parallax**. The effect of lunar parallax is to reduce the calculated altitude of the moon. The effect is at a maximum when the moon is viewed on the horizon. Figure 26 shows the geometry of the situation.

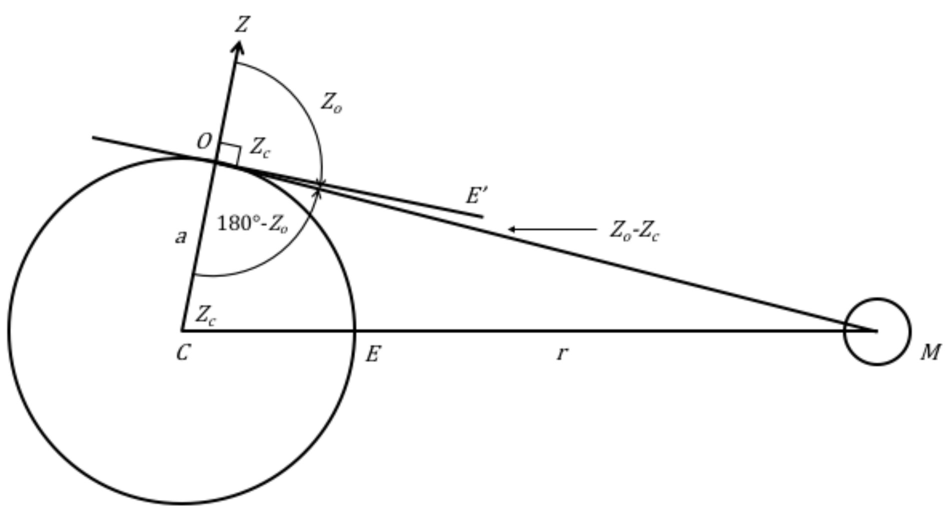


Figure 26. Aspects of parallax of the moon

The large and small circles of Figure 26 represent the Earth and moon, respectively. Let E represent the position of an observer who sees the moon at zenith whose true position on the celestial sphere is unaffected by parallax. Let O be the position of an observer, not along line CEM whose position causes the lunar parallax to be at a maximum. In such a case, the observed zenith distance, Z_o, equals 90°.

Construct a line OE' parallel to CEM. Let Z represent the zenith of the observer at position O. The calculated zenith distance (unaffected by parallax) of the moon at O is $\widehat{ZOE}$' which we shall designate by Z_c. Seen from O, the moon appears to have a greater zenith distance denoted by $\widehat{ZOM}$ due to parallax.

The apparent shift in the moon's zenith distance is $Z_o - Z_c$ that is represented by $\widehat{E'OM}$. Because $\widehat{OCM}$ corresponds to $\widehat{ZOE}$', $\widehat{OCM}$ equals Z_c. $\widehat{COM}$ is the supplement of $\widehat{ZOM}$ and therefore has a value of $180° - Z_o$ because $\widehat{E'OM}$ is an alternate interior angle to $\widehat{OMC}$, $\widehat{OMC} = Z_o - Z_c$.

Line segment CO represents the radius of the Earth, a. Line segments OM and CM represent the distances of the observer from the moon in the first case and the distance of the moon from the Earth's center in the second case. Denote these r' and r, respectively. Then, applying the sine formula from plane trigonometry yields:

$$\frac{sin(Z_o - Z_c)}{a} = \frac{sin\,Z_c}{r'}$$

or

$$sin(Z_o - Z_c) = \frac{a\, sin\, Z_c}{r'} \cong \frac{a\, sin\, Z_c}{r}$$

because $r \cong r'$.

Now $a/r = sin\, \Pi$ where Π is called the **horizontal parallax** of the moon. Parallax is a maximum when the moon is observed at the horizon. With the equatorial radius of the Earth, a, equal to $6{,}378 km$ and the mean radius of the moon's orbit, r, equal to $384{,}400 km$, we find the maximum value of parallax equal to:

$$\Pi = \sin^{-1}\left[\frac{6{,}378 km}{384{,}400 km}\right] = 57'$$

Making the substitution of $a/r = sin\, \Pi$ in the earlier equation we have:

$$sin\, (Z_o - Z_c) = sin\, \Pi\, sin\, Z_c$$

Making use of the small-angle approximation and writing Z_o, Z_c, and Π in terms of degrees yields:

$$Z_o - Z_c = \Pi\, sin\, Z_c$$

where Π has a value of $0.95°$.

Now, the observed altitude of the moon, h_o is related to the zenith distance in the following matter: $90° - h_o = Z_o$. Similarly, the moon's calculated altitude, h_c, is related to Z_c in the following fashion: $90° - h_c = Z_c$. Making these substitutions and simplifying the relationship produces a valuable result.

$$h_o = h_c - \Pi\, cos\, h_c$$

Hence, the moon's parallax is at a maximum when the moon is on the observer's horizon. Due to parallax, the moon's apparent altitude can be diminished by as much as 0.95° from the calculated value at this time.

It should be noted that is derivation is not rigorous and that the approximations in the derivation of this formula do not introduce errors larger than one-half arc minute.

9. Effects of Refraction and Semi-diameter on Rise and Set Times

From a study of physical phenomena, we know that when light exits one transparent medium and enters a different medium, it is most often redirected. This bending of a light ray is referred to as refraction. Refraction explains why a pencil partially immersed in water appears bent and accounts for a lens's ability to bring light to a focus.

Refraction due to the Earth's atmosphere can have an appreciable effect on the length of daylight, the times of rising and setting of all celestial objects, and the apparent altitudes of objects, mainly when located near the horizon where refraction is at a maximum.

When a light ray enters a medium in which its average speed is lowered (the capture and re-emission of photons by molecules or atoms accounting for the lower overall speed), its path is bent toward the normal. That is, a ray entering a medium travels through that medium along a path that approximates the direction of the normal to the surface more closely. This phenomenon is shown in Figure 27.

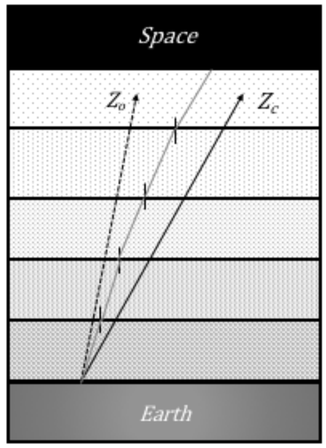


Figure 27. A parallel layers model of the atmosphere accounting for refraction.

As light enters the Earth's atmosphere from outer space, its average speed decreases. This decrease in speed of the light wave causes the path of the ray to be bent toward the vertical. If the atmosphere is considered to be planar and be composed of many parallel layers, then it is clear that a ray reaching an observer's eye appears to come from a higher elevation than the original ray. As a consequence of refraction, celestial objects appear to have a higher altitude than they do in reality. Refraction tends to increase the apparent altitude of celestial objects, especially near the horizon.

Refraction has no effect on the apparent positions of objects at the zenith (except when a frontal weather system is moving through); it has its maximum effect on objects located near the horizon. At the horizon, refraction takes on an average value of about 34 minutes of arc. Variations are on the order of tens of minutes of arc due to the changing pressure (density) and relative humidity of the atmosphere.

A detailed treatment of refraction is neither needed nor warranted for a work of this type. Suffice it to say that refraction at the horizon should be taken into account when calculating rising and setting times for celestial objects. Refraction might also be taken into account when calculating the positions of celestial objects in the sky.

The effects of refraction on the rising and setting of the sun, moon, stars, and planets can be illustrated by an analysis of Figure 28.

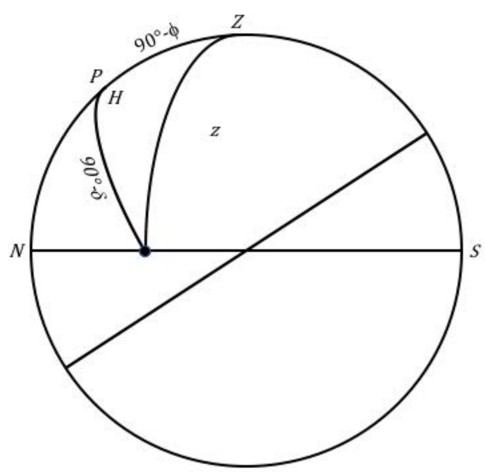


Figure 28. The relationship between solar and observer positions and the hour angle of the setting sun.

Let S represent the setting sun's position on a date when its declination is δ and observed from latitude ϕ. By applying the cosine formula to the spherical triangle PZS we have:

$$cos\ z = cos\ (90° - \phi)\ cos\ (90° - \delta) + sin\ (90° - \phi)\ sin\ (90° - \delta)\ cos\ H$$

$$cos\ z = sin\ \phi\ sin\ \delta + cos\ \phi\ cos\ \delta\ cos\ H$$

from which we arrive at:

$$H = cos^{-1}\left[\frac{cos\ z - sin\ \phi\ sin\ \delta}{cos\ \phi\ cos\ \delta}\right]$$

If sunrise or sunset is defined as the time when the sun's upper limb is on the horizon, then the sun's zenith distance is 90° plus 16' of arc, the sun's average apparent semidiameter being equal to 16' of arc. This ignores the effect of refraction that serves to reduce the apparent zenith distance of celestial objects. If refraction is taken into account, then the sun's true zenith distance at the times of rising and setting is 90° plus 16' of arc for semidiameter plus about 35' of arc for refraction for a total of approximately 90° 51'.

Concerning sunset, refraction retards the disappearance of the sun. At mid northerly latitudes on an equinox date, the time of sunset is delayed by just over 3 minutes due to the sun's semidiameter and atmospheric refraction. In polar regions, the delay can add up to hours. The time of sunrise comes earlier by a similar time interval.
Refraction, therefore, increases the length of the daylight hours by just over 6 minutes on the date of the equinoxes. Recent studies, however, have shown that the differences between the calculated and actual times of sunrise and sunset can vary by as much as three minutes due to the ever-changing refraction of the Earth's atmosphere caused by changes in pressure and relative humidity. Elevation of the observer above sea level and the horizon also can play a role.

Events involving the moon also must include apparent semidiameter and refraction. Because the moon's apparent semidiameter oscillates only slightly around the sun's mean value and the atmospheric refraction is the same, the sun and moon can be treated similarly for cases involving the need for only a good approximation.

10. Fundamentals of Celestial Navigation

Spherical astronomy can, along with a sextant, chronometer, and a nautical almanac, be used to determine one's position on the surface of Earth. An expert navigator can sometimes use a good sextant and proper technique to determine the position of a ship a sea to within 1/10 nautical mile of the true position. Unfortunately, few mariners today are familiar with celestial navigation, though Naval and Coast Guide cadets are trained in its use should newer electronic means fail for any reason.

This section has been provided to explain the basic processes associated with celestial navigation. It is not the intention of the author to create a navigator. If that is your goal, then the reader is advised look into any of a number of widely available guides to celestial navigation.

10a. Basic Concept

The basic concept behind celestial navigation is rather simple and can be readily explained with the use of two streetlamp analogies. If, for instance, a person (considered very, very short of the sake of this example – perhaps as tall as an ant) were to know the elevation of a streetlamp, then he or she could readily determine his or her position with respect to the base of the light as shown in Figure 29.

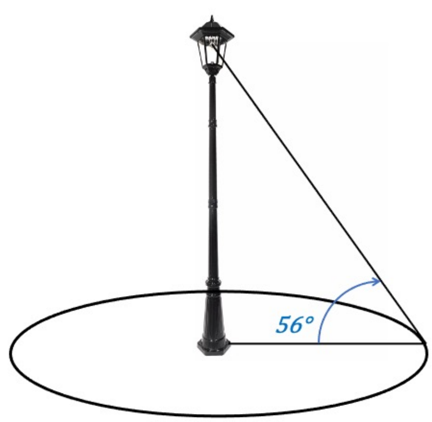


Figure 29. A streetlamp with a position circle based on a 56° elevation.

The very, very short person uses a sextant to determine that the light source (representing the sun, moon, planet, or star) is 56° above the horizon. With this information, the observer can determine his or her location with respect to the base of the streetlamp – but only to a limited extent. The observer is located on a position circle of fixed radius centered on the base of the lamp (sub-solar, sub-lunar, sub-planetary, sub-stellar point as the observed object may be). Anyone located anywhere on this position circle would see the lamp 56° above the horizon. To find the point on this circle where the observer is located will require the use of a second observation from the same location but involving a different streetlamp. This situation is illustrated in Figure 30.

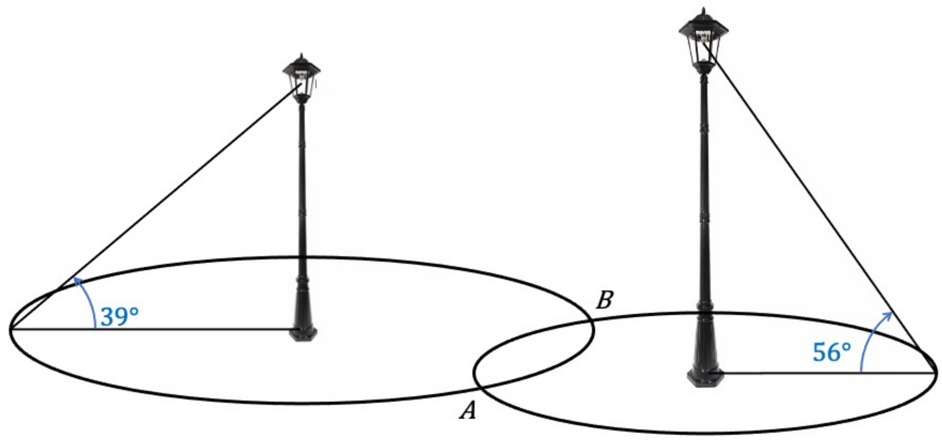


Figure 30. How two position circles reduce considerably the uncertainly in observer location.

In this second example, a new position circle is located such that the light source is 39° above the horizon. Now, this second observation restricts the observer's placement to two possible locations, either point A or point B, where the two acquisition circles intersect. Only in these two locations can the angles of elevation be 56° and 39°. A third observation using another streetlamp would uniquely determine the position of the observer, but this is rarely needed if the observer has a fair understanding of where her or she is located. (Points A and B are often hundreds of miles apart and most navigators in general know approximately where they are by the processes of dead reckoning.)

10b. Basic Procedure

How does one convert the measured elevation of celestial objects into a position on the surface of the ocean? Assume an observer position based upon course, speed, time, ocean currents from the last known position – a port say. Then, use this assumed position to calculate the expected elevation of a celestial object, h_c, using a variation of a relationship found previously:

$$sin\ h_c = sin\ \phi\ sin\ \delta + cos\ \phi\ cos\ \delta\ cos\ LHA$$

In this case LHA (the local hour angle of the star) and latitude (ϕ) will be known only approximately and these will introduce an error into the calculated position. LHA is not known with precision because $LHA\ =\ GHA - \lambda(W)$. While GHA is known with precision from a knowledge of the time at Greenwich (hence the chronometer and the nautical almanac) but $\lambda(W)$ is known only approximately.

Now, the assumed latitude of the observer and, declination and local hour angle of the star can be used with the above equation to determine the calculated elevation of the star. Comparing the calculated elevation (h_c) with the observed elevation (h_o) will tell the observer if he is closer to or farther from the substellar point than expected. As shown in Figure 31, if $h_o >\ h_c$, then the observer is closer to the substellar point than the assumed position indicates and vice versa. Each arc minute of difference is equal to a distance of one nautical mile.

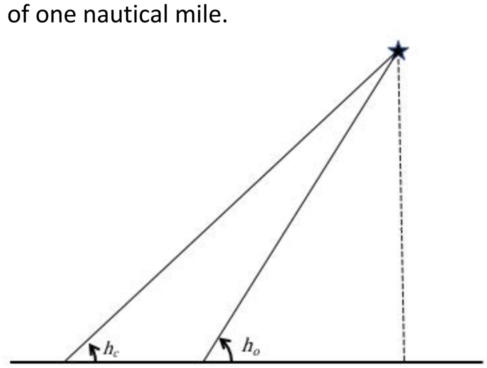


Figure 31. When $h_o >\ h_c$, the observer is closer to the substellar point than assumed.

We now need to know the azimuth of the star relative to the assumed position of the observer so we can mark off the appropriate distance in the correct direction from the assumed position. That is done with the aid of the following formula:

$$\cos A_o = \frac{\sin\delta - \sin\phi \sin h_o}{\cos\phi \cos h_o}$$

Once Az is known from A_o (recall that $Az = 360° - A_o$ if $\sin LHA < 0$), then a position line (segment of a circle of position – treated as a straight line because it is only a tiny segment of a very large circle) can be drawn on an appropriate nautical chart. The position line will be perpendicular to the line of azimuth crossing at a location $h_o - h_c$ closer (in this case) to the substellar point as shown in Figure 32.

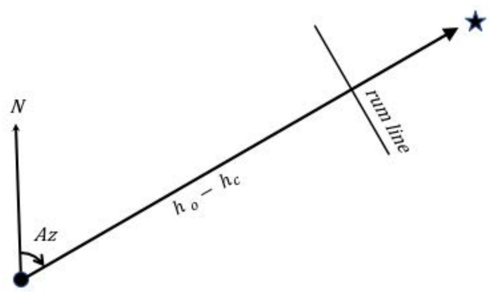


Figure 32. The azimuth line is drawn from the assumed position toward azimuth of the substellar point. If $h_o > h_c$, then the position line is drawn closer to the substellar point a distance in nautical miles equal $h_o - h_c$ (expressed in arc minutes). The converse is also true. The true position of the observer is somewhere along this position line.

A fix is then taken on a second star and the value of $h_o - h_c$ computed. A second line of azimuth is then drawn on a nautical chart from the assumed position to the second substellar point and a second position line drawn. The observer must also be located somewhere on this second position line. Where the position lines cross shows the true location of the observer as shown in Figure 33.

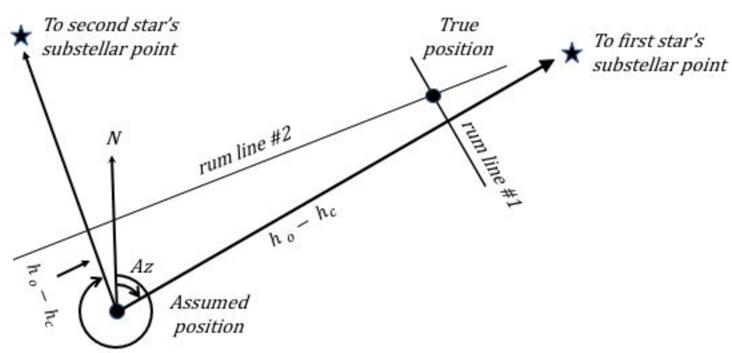


Figure 33. Where the two position lines cross indicates the true position of the observer.

10c. Corrections to Sextant Observations

Sextants are used to measure the apparent elevation of a celestial object above the horizon. Failure to take necessary correction to apparent elevation can lead to large errors in position. A detailed study of the sextant is needed to fully understand how the process works as well as the corrections, but the corrections are summarized here as follows:

- **Index Error** – A sextant consists of two mirrors that should indicate an elevation of zero degrees when sighting an object on the horizon. The mirrors can be aligned so that the measured elevation is indeed zero, but mirror alignment can sometimes be a tricky process and the mirrors can be jiggled when the sextant is stored and removed for use. Rather than try to correct the mirrors with each use of the sextant, the deviation from zero degrees when viewing something on the horizon. This error typically can amount as much as about 5 arc minutes and therefore cannot be ignored.
- **Semidiameter** – If the upper or lower limbs of the solar or lunar disks are used for location determination, then the semidiameter must either be subtracted (in the event of using the upper edge) or added (in the case of using the lower

edge) to correct the observation. Solar and lunar declinations are listed in the nautical almanac for these bodies' centers. Semidiameters are tabulated in the Nautical Almanac for every day of the year.

- **Lunar Parallax** – Of all the celestial bodies, only the moon is sufficiently close to the Earth to demonstrate considerable parallax. A correction for lunar parallax must also be made because parallax will serve to change the apparent position of the moon in the sky.
- **Dip of the Horizon** – The observer's height above sea level causes the horizon to have a zenith distance of more than 90°. This excess is called the dip of the horizon. Geometrically speaking, this dip (and the distance to the horizon) can be easily shown to have a standard mathematical form. Unfortunately, atmospheric refraction due to the layer between the ocean surface and the height of the sextant horizontal optical axis above the ocean surface complicates the matter considerably. Typically, the dip of the horizon is quite different when the temperature of the air decreases with increasing distance or suffers a temperature inversion (temperature of the air increases with increasing distance above the ocean). A general rule for dip of the horizon is given that will have to suffice for the here now. $\theta = 1.92\sqrt{h}$, where θ is expressed in arc minutes and h is the distance above the ocean surface expressed in meters. For instance, if a sailor's eye is $8m$ above the surface of the ocean, then $\theta = 1.92\sqrt{8} = 5.43'$ approximately. Failure to correct for this amount of dip in the horizon can affect the ships determined position to be in error by more than 5 nautical miles. (1 arc minute corresponds to 1 nautical mile = $6080ft$ at the equator.
- **Atmospheric Refraction** – Atmospheric refraction causes objects to appear higher in the sky than they actually are. That's why sunrises occur earlier and sunsets later than would be expected if there were no atmosphere. As noted in the section on refraction, refraction can amount to as much as $35'$ at the horizon but diminishes to $0'$ at the zenith. A functional formula for refraction (there are many variations) is as follows: $R = cot(h + 7.31/(h + 4.4))$ where h is the true elevation in degrees, and R is expressed in arc minutes). This formula gives no consideration to the components of pressure, relative humidly, and temperature. It will suffice, however, for the purpose of this illustration.

10d. Position at Sea – USCG Eagle

United States Coast Guard cutter Eagle (an 1877 tall ship) is used for training in basic seamanship. One of the skills taught to cadettes is celestial navigation using a sextant and chronometer (an accurate clock). The cutter is approaching its home port in New London, Connecticut. The date is June 20, 2020. The stars Vega and Arcturus are "shot" during evening twilight (before the end of nautical twilight when the horizon is too dark to see) to determine the ship's position. The assumed position based on dead reckoning is as follows:

$$\phi = 42.0°\ N$$

$$\lambda(W) = 72.5°$$

The following data are collected in relation to each star whose positions have been precessed to the date J2020.5:

Star: $\boldsymbol{Vega}$	Star: $\boldsymbol{Arcturus}$
Time of shot: $9{:}25\ PM$	Time of shot: $9{:}30\ PM$
GHA: $00h\ 46.2'W$	GHA: $05h\ 07.2'W$
$\alpha = 18h\ 37.6m$ (J2020.5)	$\alpha = 14h\ 16.6m$ (J2020.5)
$LHA = GHA - \lambda(W) = 04h\ 02.2'E$	$LHA = GHA - \lambda(W) = 00h\ 23.8'W$
$\delta = +38°\ 48.1'$ (J2020.5)	$\delta = +19°\ 05.5'$ (J2020.5)

Figure 31 is used to derive the following relationships:

$$sin\ h_c = sin\ \phi\ sin\ \delta + cos\ \phi\ cos\ \delta\ cos\ LHA$$

$$cos\ A_o = \frac{sin\ \delta - sin\ \phi\ sin\ h}{cos\ \phi\ cos\ h}$$

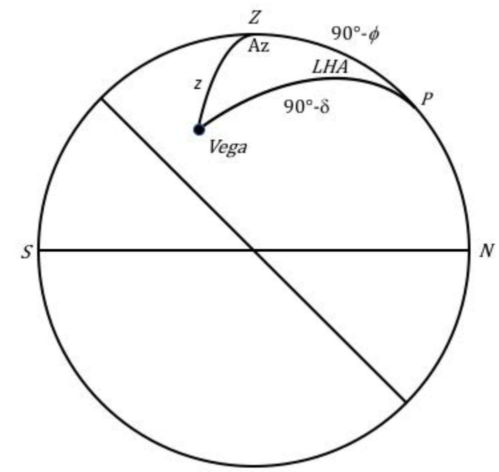


Figure 31. The spherical triangle used to calculate elevation and azimuth of celestial object based on assumed position.

The following data are then generated for h_c and Az using the assumed position. Included are the corrected sextant observations yielding h_o. Also given is the difference $h_o - h_c$

Star: $\boldsymbol{Vega}$ | Star: $\boldsymbol{Arcturus}$

$$h_o = 44° \, 50.5'$$
$$h_c = 44° \, 28.2'$$

$h_o - h_c = +22.3$' (toward star)

$$Az = 73° \, 08.8m$$

$$h_o = 66° \, 14.8'$$
$$h_c = 66° \, 36.5'$$

$h_o - h_c = -21.7$' (opposite star)

$$Az = 194° \, 05.8'$$

Figure 32 shows the assumed position and the stellar azimuths and position lines at the appropriate distances and directions from the substellar points. The location where the position lines cross is the actual position of the ship. The latitude and longitude of the actual position are taken from the gridlines on the nautical chart. The distance between the two locations is 22.5' or 22.5 nautical miles.

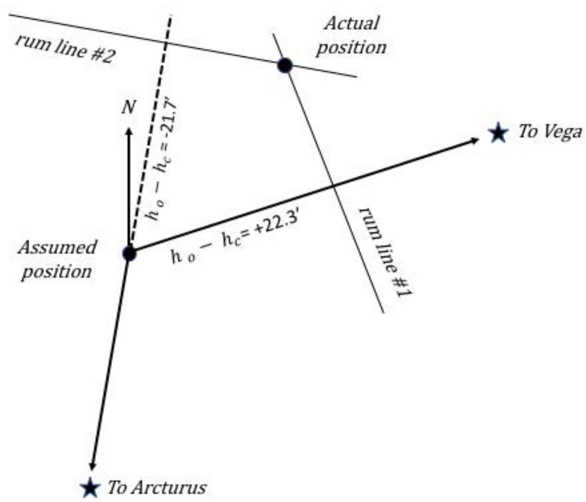


Figure 32. Map solution for the actual position of the Eagle.

11. Areas on the Celestial Sphere

The area on the surface of a sphere is frequently given as a solid angle. The unit of solid angular measure called the **steradian**. The surface area of a sphere is 4π steradians, where the steradian is defined as follows:

$$1\ steradian = \left(\frac{180°}{\pi}\right)^2$$

Because each steradian intercepts approximately 3,283 square degrees, the sky that has 4π steradians contains approximately $41,253$ square degrees.

What would be the area, A, of a spherical polygon on the surface of a sphere? From the study of spherical trigonometry we know that the area of such a polygon *in steradians* is given by the following formula:

$$A = \frac{\pi r^2 E}{180°}$$

where r is the radius of the sphere (set equal to unity) and E is the **spherical excess** defined by the following relationship:

$$E = T - 180°(n-2)$$

In this relationship, T equals the total of all angles in the polygon, and n equals the number of sides.
For a spherical triangle ($n = 3$) on the surface of the unit sphere ($r = 1$) the area inscribed in *steradians* is found to be:

$$A = \frac{\pi}{180°}(A + B + C - 180°)$$

where A, B, and C are the angles of the spherical triangle expressed in degrees.

For a polar triangle in which all sides and angles equal 90°, the area is found to be $\pi/2$ steradians. This equals one-eighth of the total 4π steradian sphere that is the expected result.

If we choose to rewrite A in terms of *square degrees*, the above equation becomes:

$$A = \frac{180°}{\pi}(A + B + C - 180°)$$

If the above formula is now applied to determine the number of square degrees present in, say, the Summer Triangle, that region of the sky bordered by great circle arcs extending from Vega to Altair to Deneb and back again to Vega, we must first calculate the angular separations between each pair of stars by applying the cosine formula:

$$cos\ d = sin\ \delta_1\ sin\ \delta_2 + cos\ \delta_1\ cos\ \delta_2\ cos\ (\Delta\alpha)$$

Results of these calculations are to be found below:

Altair-Vega ($\widehat{AV}$)	Altair-Deneb ($\widehat{AD}$)	Deneb-Vega ($\widehat{DV}$)
$34.195°$	$38.014°$	$23.847°$

Once the angular separations between the stars are found, the vertices of the Summer Triangle, A, B, and C, can be found by reapplying the cosine formula:

$$cos\ A = \frac{cos\ \widehat{AD} - cos\ \widehat{AV}\ cos\ \widehat{DV}}{sin\ \widehat{AV}\ \sin \widehat{DV}}$$

where A equals the vertex of an angle as seen from Altair and $\widehat{AD}$, $\widehat{AV}$, and $\widehat{DV}$ are the angular separations between the three stars found previously. Formulas for the remaining vertices are similar. The three resulting vertices are given below:

Vega-Deneb-Altair	Deneb-Altair-Vega	Altair-Vega-Deneb
$64.669°$	$40.557°$	$82.071°$

Inserting these values into the equation for the area of a spherical triangle *in degrees* given above values yields an area for the Summer Triangle equal to some 418.09 square degrees.

